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A Formal State-Property Duality in Quantum Logic*, †

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Abstract. This paper presents a formalization of the state-property duality in quantum physics. On the side of properties, Piron shows that Piron lattices, originally called irreducible propositional systems, capture the essential structure formed by the testable properties of quantum systems. On the side of states, we define quantum Kripke frames to capture the essential structure formed by the states of quantum systems under the non-orthogonality relation. Moreover, we define linear morphisms between Piron lattices, and then organize the class of Piron lattices into a category. We also define continuous homomorphisms between quantum Kripke frames, and then organize the class of quantum Kripke frames into a category. Finally, we will show a duality, in the sense of category theory, between the category of Piron lattices and the category of quantum Kripke frames, and thus capture the conceptual state-property duality in quantum physics in a mathematical language. This formal duality, connecting algebraic structures with relational structures, will be helpful in the study of logics of quantum physics.

1 Introduction

States and properties are two important theoretical notions in physics. Roughly speaking, for a physical system, a state is a complete specification, and a property is a feature that can be tested by experiments. Intuitively, states and properties are two different perspectives of modelling a physical system which form a duality: every

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†The object part of this duality result, i.e. the correspondence between Piron lattices and quantum Kripke frames, is proved in an indirect way in my PhD thesis [14] (Theorem 2.7.23) finished at the Institute for Logic, Language and Computation (ILLC), University of Amsterdam, in 2015. I am very grateful to my supervisors, Prof. Johan van Benthem, Dr. Alexandru Baltag and Prof. Sonja Smets, for their supervision and the detailed comments on my thesis which are invaluable in the writing of this paper. I am also very grateful to the members in my PhD committee, Dr. Nick Bezhanishvili, Prof. Robert Goldblatt, Prof. John Harding, Prof. Yde Venema, Prof. Ronald de Wolf and Prof. Mingsheng Ying, for their comments on my thesis which are very helpful in writing this paper. The correspondence was also presented in many workshops and seminars, and I thank the audiences for their helpful comments. For the details, please refer to the Acknowledgement of [14]. The full duality result was presented in the workshop associated with my PhD defence, and I am very grateful to the audience for their interesting comments and discussion. Finally, I thank very much the reviewers of this paper for their patience and time in reading and checking the tedious proofs and for their helpful comments.

state can be associated with the set of properties which the system has in the state; and, every property can be associated with the set of states in which the system has the property. In this paper, we try to formalize this conceptual state-property duality in quantum physics using the mathematical language of category theory.[1] Like Stone duality for classical logic, Jónsson-Tarski duality for modal logic and Esakia duality for intuitionistic logic, such a formal duality will be useful for the study of quantum logic.[2] In the mathematical aspect, this paper is similar to [10] by Moore, which also presents a formal state-property duality. However, the mathematical structures involved in this paper are more specific and connect more closely to quantum theory than those in [10], which I am about to explain.

On the side of properties, it was the great observation of Birkhoff and von Neumann in [4] that, according to quantum theory, only some special collections of states of a quantum system correspond to properties which can be tested by experiments. Such properties are now called testable properties, and they form a lattice which is modelled by a Hilbert lattice, i.e. the lattice of closed linear subspaces of a Hilbert space. It was also a great observation in [4] that in such a lattice distributivity between meets and joins fails. The logical study of Hilbert lattices and their generalizations then becomes an active research field which is called quantum logic. A milestone in this field is a theorem of Piron published in his PhD thesis ([12])[3] in 1964. Roughly speaking, this theorem shows that the essential structure of a Hilbert lattice, and thus of a lattice formed by the testable properties of a quantum system, is captured in the definition of a Piron lattice, which, originally called an irreducible propositional system in [13], is a mild and purely lattice-theoretic generalization of a Hilbert lattice. Based on Piron's theorem, in this paper we take Piron lattices as mathematical models of the structure formed by (testable) properties.

On the side of states, the works of Dishkant ([6]), Goldblatt ([7]) and Hedlíková and Pulmannová ([9]) show that the properties of the (non-)orthogonality relation between the states of a quantum system play an important role in the study of quantum logic. According to quantum theory, two states are non-orthogonal, if a measurement can trigger the system to change from one of the states to the other. Hence a model of a quantum system based on the non-orthogonality relation only employs a minor philosophical assumption: measurements may change the state of the system. In this paper we use a kind of Kripke frames, called quantum Kripke frames, to model the structure formed by states under the non-orthogonality relation. Quantum Kripke frames are proposed in [14], where their structure is studied in detail and their significance in physics is demonstrated (in particular, Corollary 2.5.6 and Corollary 2.7.19

[1] For the notions and results in category theory used in this paper, please refer to the textbook [1].

[2] For dualities and their application in the study of logic, please refer to [11].

[3] Piron's thesis is in French. His theorem was first published in English in the book [13], together with a detailed discussion of its significance and application in physics.

in [14]).

Moreover, for the categorical arrows which connect two quantum structures in the same category, we use linear morphisms between Piron lattices and continuous homomorphisms between quantum Kripke frames, both of which are mild generalizations of bounded linear maps between Hilbert spaces. In quantum theory, such linear maps play a crucial role. Unitary operators describing evolution, projectors describing tests of properties and Hermitian operators describing some important observables are all bounded linear maps. Besides, the tensor product of two Hilbert spaces, which describes quantum entanglement, can also be constructed from bounded linear maps. Therefore, the arrows in our categories have significance in physics. In my knowledge, linear morphisms between Piron lattices are defined for the first time in this paper, although some generalizations of them exist in the literature, e.g. [10]. Continuous homomorphisms between quantum Kripke frames are proposed in [14], where their properties are studied in detail and their significance in physics is demonstrated (in particular, Corollary 3.1.14 in [14]).

The organization of this paper is as follows. In Section 2 we define the category $\mathbb{L}$ of Piron lattices and the category $\mathbb{F}$ of quantum Kripke frames. In Section 3 we define two functors $\mathbf{F} : \mathbb{L}^{op} \to \mathbb{F}$ and $\mathbf{G} : \mathbb{F}^{op} \to \mathbb{L}$. In Section 4 we define two natural isomorphisms $\tau : 1_{\mathbb{L}} \to \mathbf{G} \circ \mathbf{F}$ and $\eta : 1_{\mathbb{F}} \to \mathbf{F} \circ \mathbf{G}$ which make $(\mathbf{F}, \mathbf{G})$ form a duality. In each section, main results are called theorems, and the results directly connecting to some main result are called propositions; on the way we also prove some technical and useful lemmas.

2 The Categories

In this section, we will formally define the category $\mathbb{L}$ of Piron lattices and the category $\mathbb{F}$ of quantum Kripke frames.

2.1 The Category $\mathbb{L}$ of Piron Lattices

Piron lattices, the objects of $\mathbb{L}$, were first defined by Piron in [12], where they were called irreducible propositional systems. They are mathematical models of the structure formed by testable properties of quantum systems, where each testable property is modelled by an element in a Piron lattice.

In quantum mechanics, a Piron lattice is a generalization of a Hilbert lattice. To be precise, given a Hilbert space $\mathcal{H}$ over $\mathbb{C}$, let $\mathcal{L}(\mathcal{H})$ be the set of closed linear subspaces of $\mathcal{H}$, $\subseteq$ the subset relation and $(\cdot)^{\perp} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ the orthocomplement operation of closed linear subspaces. One can show that the tuple $\mathfrak{L}_{\mathcal{H}} = (\mathcal{L}(\mathcal{H}), \subseteq, (\cdot)^{\perp})$ forms a lattice called a Hilbert lattice, and a Piron lattice is a generalization of such a lattice.

Formally, the notion of a Piron lattice is defined as follows:

Definition 1 A *Piron lattice* $\mathfrak{L}$ is a tuple $(L, \leq, (\cdot)^{\perp})$, where L is a non-empty set, $\leq \subseteq L \times L$ and $(\cdot)^{\perp} : L \to L$ is a function such that the following conditions hold:

1. $(L, \leq)$ is a lattice;
2. *completeness*: each $A \subseteq L$ has an infimum $\bigwedge A$ and a supremum $\bigvee A$ in L;
3. *boundedness*: there are *distinct* $O, I \in L$ such that $O \leq a \leq I$ for each $a \in L$;
4. *orthocomplementation*: $(\cdot)^{\perp}$ is an orthocomplementation, i.e. for any $a, b \in L$,
   (a) $a \wedge a^{\perp} = O$ and $a \vee a^{\perp} = I$;
   (b) $a \leq b$ implies that $b^{\perp} \leq a^{\perp}$;
   (c) $a^{\perp\perp} = a$;
5. *atomicity*: for each $a \in L \setminus \{O\}$, there is a $p \in At(\mathfrak{L})$[4] such that $p \leq a$;
6. *weak modularity*: for any $a, b \in L$, $a \leq b$ implies that $a = b \wedge (a \vee b^{\perp})$;
7. *covering law*: if $p \in At(\mathfrak{L})$ and $a \in L$ satisfy $p \wedge a = O$, $(p \vee a) \wedge a^{\perp} \in At(\mathfrak{L})$;
8. *superposition principle*: for any $p, q \in At(\mathfrak{L})$ satisfying $p \neq q$, there is an $r \in At(\mathfrak{L}) \setminus \{p, q\}$ such that $p \vee q = p \vee r = q \vee r$.

This definition is a slight modification of, though equivalent to, that in [13]. It is from [2], which also contains an explanation for such a modification.

The following lemma collects some well-known facts about Piron lattices.

Lemma 2 Let $\mathfrak{L} = (L, \leq, (\cdot)^{\perp})$ be a Piron lattice.

1. De Morgan's Law: $(a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp}$ and $(a \wedge b)^{\perp} = a^{\perp} \vee b^{\perp}$, for any $a, b \in L$.
2. For any $a \in L$, $a \in At(\mathfrak{L}) \Leftrightarrow a^{\perp} \in coAt(\mathfrak{L})$[5], and $a \in coAt(\mathfrak{L}) \Leftrightarrow a^{\perp} \in At(\mathfrak{L})$.
3. If $a, b \in L$ satisfy $a \leq b$, $b = a \vee (a^{\perp} \wedge b)$.
4. $\mathfrak{L}$ is atomistic, i.e. for any $a \in L$, $a = \bigvee [\![a]\!]$, where $[\![a]\!] \stackrel{\text{def}}{=} \{p \in At(\mathfrak{L}) \mid p \leq a\}$.

Proof The proofs use boundedness and orthocomplementation, and are easy. 3 needs weak modularity; 4 needs weak modularity, completeness and atomicity. □

Next we define the notion of a linear morphism from one Piron lattice to another (not necessarily distinct). This notion is a generalization of a bounded linear map between Hilbert spaces. To be precise, let f be a bounded linear map from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$, and define a function $\mathcal{L}(f) : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2) :: V \mapsto f[V]$. One can show that $\mathcal{L}(f)$ is a linear morphism, in the sense of the following formal definition:

[4]In this paper I use $At(\mathfrak{L})$ to denote the set of atoms of a bounded lattice $\mathfrak{L} = (L, \leq)$. An *atom* of $\mathfrak{L}$ is a $p \in L$ such that $p \neq O$ and, for each $a \in L$, $O \leq a \leq p$ implies that either $a = O$ or $a = p$.

[5]In this paper $coAt(\mathfrak{L})$ denotes the set of coatoms of a bounded lattice $\mathfrak{L} = (L, \leq)$. A *coatom* of $\mathfrak{L}$ is a $p' \in L$ such that $p' \neq I$ and, for each $b \in L$, $p' \leq b \leq I$ implies that either $b = I$ or $b = p'$.

Definition 3 A function $h : L_1 \to L_2$ is a *linear morphism*, or an $\mathbb{L}$-*morphism*, from a Piron lattice $\mathfrak{L}_1 = (L_1, \leq_1, (\cdot)^{\perp_1})$ to a Piron lattice $\mathfrak{L}_2 = (L_2, \leq_2, (\cdot)^{\perp_2})$, if the following conditions hold:

1. *meet preservation*: $h(\bigwedge_1 A) = \bigwedge_2 h[A]$ for any $A \subseteq L_1$;
2. *Moore's condition*: if $p_2 \in At(\mathfrak{L}_2)$, $p_2 \leq_2 h(p_1)$ for some $p_1 \in At(\mathfrak{L}_1)$;
3. *dual Moore's condition*: if $a_1 \in coAt(\mathfrak{L}_1)$, $a_2 \leq_2 h(a_1)$ for some $a_2 \in coAt(\mathfrak{L}_2)$.

Note that each linear morphism h is monotone, because by meet preservation

$$a_1 \leq_1 b_1 \Rightarrow a_1 = a_1 \wedge b_1 \Rightarrow h(a_1) = h(a_1 \wedge b_1) = h(a_1) \wedge h(b_1) \Rightarrow h(a_1) \leq_2 h(b_1)$$

Now we state and prove the main result of this subsection.

Theorem 4 Piron lattices equipped with linear morphisms form a category $\mathbb{L}$.

Proof We define arrow composition to be function composition, and identity arrows to be identity functions, i.e. $id_{\mathfrak{L}} = Id_L$, for each Piron lattice $\mathfrak{L} = (L, \leq, (\cdot)^{\perp})$.

It is obvious that identity functions are linear morphisms, and, for each linear morphism $h : \mathfrak{L}_1 \to \mathfrak{L}_2$, $id_{\mathfrak{L}_2} \circ h = Id_{L_2} \circ h = h = h \circ Id_{L_1} = h \circ id_{\mathfrak{L}_1}$.

Moreover, function composition satisfies associativity. It remains to show that, for two linear morphisms $h : \mathfrak{L}_1 \to \mathfrak{L}_2$ and $g : \mathfrak{L}_2 \to \mathfrak{L}_3$, $g \circ h$ is a linear morphism.

For meet preservation, for each $A \subseteq L_1$,

$$(g \circ h)(\bigwedge_1 A) = g\big(h(\bigwedge_1 A)\big) = g\,(\bigwedge_2 h[A]) = \bigwedge_3 g\,[h[A]] = \bigwedge_3 (g \circ h)[A]$$

For Moore's condition, for each $p_3 \in At(\mathfrak{L}_3)$, since g is a linear morphism, there is a $p_2 \in At(\mathfrak{L}_2)$ such that $p_3 \leq_3 g(p_2)$. For $p_2 \in At(\mathfrak{L}_2)$, since h is a linear morphism, there is a $p_1 \in At(\mathfrak{L}_1)$ such that $p_2 \leq_2 h(p_1)$. By monotonicity $g(p_2) \leq_3 g(h(p_1)) = (g \circ h)(p_1)$. Therefore, $p_1 \in At(\mathfrak{L}_1)$ is such that $p_3 \leq_3 (g \circ h)(p_1)$.

For the dual Moore's condition, for each $a_1 \in coAt(\mathfrak{L}_1)$, since h is a linear morphism, there is an $a_2 \in coAt(\mathfrak{L}_2)$ such that $a_2 \leq_2 h(a_1)$. By monotonicity $g(a_2) \leq_3 g(h(a_1)) = (g \circ h)(a_1)$. For $a_2 \in coAt(\mathfrak{L}_2)$, since g is a linear morphism, there is an $a_3 \in coAt(\mathfrak{L}_3)$ such that $a_3 \leq_3 g(a_2)$. Therefore, $a_3 \in coAt(\mathfrak{L}_3)$ is such that $a_3 \leq_3 (g \circ h)(a_1)$. □

2.2 The Category $\mathbb{F}$ of Quantum Kripke Frames

Quantum Kripke frames, the objects of $\mathbb{F}$, were first defined in [14]. They are mathematical models of the structure formed by the states of quantum systems, where each state is modelled by an element in a quantum Kripke frame.

In quantum mechanics, a quantum system is described by a Hilbert space $\mathcal{H}$ over $\mathbb{C}$ in such a way that the states of the system are modelled by the one-dimensional subspaces of $\mathcal{H}$. We denote by $\Sigma(\mathcal{H})$ the set of one-dimensional subspaces of $\mathcal{H}$, and define a binary relation $\rightarrow_{\mathcal{H}}$ on $\Sigma(\mathcal{H})$ such that, for any $s, t \in \Sigma(\mathcal{H})$, $s \rightarrow_{\mathcal{H}} t$, if and only if there are $\mathbf{s} \in s$ and $\mathbf{t} \in t$ such that $\langle \mathbf{s}, \mathbf{t} \rangle \neq 0$. A quantum Kripke frame is a generalization of the tuple $\mathfrak{F}_{\mathcal{H}} = (\Sigma(\mathcal{H}), \rightarrow_{\mathcal{H}})$.[6]

Now we proceed gradually to the formal definition of quantum Kripke frames. We start from the notion of Kripke frames.

Definition 5 A *Kripke frame* is a tuple $\mathfrak{F} = (\Sigma, \rightarrow)$ in which Σ is a non-empty set and $\rightarrow \subseteq \Sigma \times \Sigma$.

The following are some notions and notations in a Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$:

- If $(s, t) \in \rightarrow$, we call that s and t are *non-orthogonal* and write $s \rightarrow t$.
- If $(s, t) \notin \rightarrow$, we call that s and t are *orthogonal* and write $s \not\rightarrow t$.
- $\sim P \stackrel{\text{def}}{=} \{s \in \Sigma \mid s \not\rightarrow t \text{ for all } t \in P\}$ is called the *orthocomplement* of $P \subseteq \Sigma$.
- $P \subseteq \Sigma$ is *bi-orthogonally closed*, if $\sim\sim P = P$.
- $\mathcal{L}_{\mathfrak{F}} \stackrel{\text{def}}{=} \{P \subseteq \Sigma \mid \sim\sim P = P\}$.
- $s, t \in \Sigma$ are *indistinguishable with respect to* $P \subseteq \Sigma$, denoted by $s \approx_P t$, if for every $x \in P$, $s \rightarrow x \Leftrightarrow t \rightarrow x$.
- $t \in \Sigma$ is *an approximation of* $s \in \Sigma$ *in* $P \subseteq \Sigma$, if $t \in P$ and $s \approx_P t$.

Now we are ready to define formally the notion of quantum Kripke frames.

Definition 6 A *quantum Kripke frame* is a Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$ satisfying:

1. *reflexivity:* $s \rightarrow s$, for every $s \in \Sigma$;
2. *symmetry:* $s \rightarrow t$ implies that $t \rightarrow s$, for any $s, t \in \Sigma$;
3. *separation:* for any $s, t \in \Sigma$, if $s \neq t$, $w \rightarrow s$ and $w \not\rightarrow t$ for some $w \in \Sigma$;
4. *superposition:* for any $s, t \in \Sigma$, there is a $w \in \Sigma$ such that $w \rightarrow s$ and $w \rightarrow t$;
5. *existence of approximation:* for any $P \in \mathcal{L}_{\mathfrak{F}}$ and $s \in \Sigma$, if $s \notin \sim P$, there is an $s' \in \Sigma$, which is an approximation of s in P, i.e. $s' \in P$ and $s \approx_P s'$.

Here we prove two useful lemmas about quantum Kripke frames. One of them is about some basic facts about orthocomplements.

Lemma 7 In a quantum Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$, for any $P, Q \subseteq \Sigma$,

1. $\sim\emptyset = \Sigma$ and $\sim\Sigma = \emptyset$, and thus $\sim\sim\emptyset = \emptyset$ and $\sim\sim\Sigma = \Sigma$, i.e. $\emptyset, \Sigma \in \mathcal{L}_{\mathfrak{F}}$;
2. $P \cap \sim P = \emptyset$;
3. $P \subseteq Q$ implies that $\sim Q \subseteq \sim P$;
4. $P \subseteq \sim\sim P$.

[6] Reflexivity and symmetry follow from definite positiveness and conjugate symmetry of the inner product, respectively; separation follows from the orthogonalization trick in Gram-Schmidt's Theorem; superposition can be proved by taking an appropriate linear combination of vectors; and existence of approximation follows from the orthogonal decomposition theorem.

Proof The proof only uses reflexivity and symmetry, and is easy. □

The other one implies that $\mathcal{L}_{\mathfrak{F}}$ is closed under intersection and orthocomplement.

Lemma 8 In a quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$,

1. $\bigcap_{i\in I} P_i \in \mathcal{L}_{\mathfrak{F}}$, if $P_i \in \mathcal{L}_{\mathfrak{F}}$ for each $i \in I$;
2. $\sim P \in \mathcal{L}_{\mathfrak{F}}$, for each $P \subseteq \Sigma$.

Proof For *1*, when $I = \emptyset$, we use the convention that $\bigcap_{i\in I} P_i = \Sigma$, so by Lemma 7 $\bigcap_{i\in I} P_i \in \mathcal{L}_{\mathfrak{F}}$. When $I \neq \emptyset$, by Lemma 7 $\bigcap_{i\in I} P_i \subseteq \sim\sim\bigcap_{i\in I} P_i$. For the other direction, let $i \in I$ be arbitrary. Since $\bigcap_{i\in I} P_i \subseteq P_i$, $\sim\sim\bigcap_{i\in I} P_i \subseteq \sim\sim P_i = P_i$ by Lemma 7 and $P_i \in \mathcal{L}_{\mathfrak{F}}$. For $i \in I$ is arbitrary, $\sim\sim\bigcap_{i\in I} P_i \subseteq \bigcap_{i\in I} P_i$.

For *2*, by Lemma 7 $\sim P \subseteq \sim\sim\sim P$ and $P \subseteq \sim\sim P$. Applying Lemma 7 to the latter we get $\sim\sim\sim P \subseteq \sim P$. Therefore, $\sim P = \sim\sim\sim P$, and thus $\sim P \in \mathcal{L}_{\mathfrak{F}}$. □

Next we define the notion of a continuous homomorphism from a quantum Kripke frame to another (not necessarily distinct). It is also a generalization of a bounded linear map between Hilbert spaces. To be precise, let $f : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear map from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$, and define a partial function $\mathcal{P}(f) : \Sigma(\mathcal{H}_1) \dashrightarrow \Sigma(\mathcal{H}_2)$ in the following way:

$$\mathcal{P}(f)(s) = \begin{cases} f[s], & \text{if } f[s] \neq \{\mathbf{0}_2\}; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Then $\mathcal{P}(f)$ is a continuous homomorphism, in the sense of the following definition:

Definition 9 A partial function $F : \Sigma_1 \dashrightarrow \Sigma_2$ is a *continuous homomorphism*, or an $\mathbb{F}$-*morphism*, from a quantum Kripke frame $\mathfrak{F}_1 = (\Sigma_1, \to_1)$ to $\mathfrak{F}_2 = (\Sigma_2, \to_2)$, if the following condition holds: for any $w_1 \in \Sigma_1$ and $w_2, t_2 \in \Sigma_2$, if $F(w_1) = w_2$ and $w_2 \to_2 t_2$, there is a $t_1 \in \Sigma_1$ such that (t_1, t_2) satisfies:

(Ad)$_F$ for every $s_1 \in \Sigma_1$, $t_1 \to_1 s_1 \iff \big(F(s_1)$ is defined and $t_2 \to_2 F(s_1)\big)$

Theorem 10 Quantum Kripke frames equipped with continuous homomorphisms form a category $\mathbb{F}$.

Proof We define arrow composition to be the composition of partial functions, and identity arrows to be identity functions, i.e. $id_{\mathfrak{F}} = Id_{\Sigma}$ for each $\mathfrak{F} = (\Sigma, \to)$.

Note that identity functions are continuous homomorphisms, because the defining condition boils down to a trivial one: for any $w, t \in \Sigma$, if $w \to t$, there is an $s \in \Sigma$ such that, for every $r \in \Sigma$, $s \to r \Leftrightarrow t \to r$. Moreover, for each continuous homomorphism $F : \mathfrak{F}_1 \to \mathfrak{F}_2$, $id_{\mathfrak{F}_2} \circ F = Id_{\Sigma_2} \circ F = F = F \circ Id_{\Sigma_1} = F \circ id_{\mathfrak{F}_1}$.

Note that the composition of two partial functions is a partial function, and such a composition satisfies associativity. It remains to show that, for two continuous homomorphisms $F : \mathfrak{F}_1 \to \mathfrak{F}_2$ and $G : \mathfrak{F}_2 \to \mathfrak{F}_3$, $G \circ F$ is a continuous homomorphism.

Assume that $w_1 \in \Sigma_1$ and $w_3, t_3 \in \Sigma_3$ satisfy $(G \circ F)(w_1) = w_3$ and $w_3 \to_3 t_3$. Since $(G \circ F)(w_1) = w_3$, there is a $w_2 \in \Sigma_2$ such that $F(w_1) = w_2$ and $G(w_2) = w_3$. Since $G(w_2) = w_3$ and $w_3 \to_3 t_3$, by definition there is a $t_2 \in \Sigma_2$ such that (t_2, t_3) satisfies the following:

(Ad)$_G$ for every $s_2 \in \Sigma_2$, $t_2 \to_2 s_2 \iff \big(G(s_2)$ is defined and $t_3 \to_3 G(s_2)\big)$

Applying this to $w_2 \in \Sigma_2$, for $G(w_2) = w_3 \to_3 t_3$, $w_2 \to_2 t_2$. Since $F(w_1) = w_2$ and $w_2 \to_2 t_2$, by definition there is a $t_1 \in \Sigma_1$ such that (t_1, t_2) satisfies:

(Ad)$_F$ for every $s_1 \in \Sigma_1$, $t_1 \to_1 s_1 \iff \big(F(s_1)$ is defined and $t_2 \to_2 F(s_1)\big)$

Note that, for each $s_1 \in \Sigma_1$,

$$\begin{aligned} t_1 \to s_1 &\Leftrightarrow F(s_1) \text{ is defined and } t_2 \to_2 F(s_1) \\ &\Leftrightarrow F(s_1) \text{ is defined and } G(F(s_1)) \text{ is defined and } t_3 \to_3 G(F(s_1)) \\ &\Leftrightarrow (G \circ F)(s_1) \text{ is defined and } t_3 \to_3 (G \circ F)(s_1) \end{aligned}$$

Hence (t_1, t_3) satisfies (Ad)$_{G \circ F}$; so $G \circ F$ is a continuous homomorphism. □

3 The Functors

In this section, we give the definitions of two functors $\mathbf{F} : \mathbb{L}^{op} \to \mathbb{F}$ and $\mathbf{G} : \mathbb{F}^{op} \to \mathbb{L}$, and show that they are well defined.

3.1 From Piron Lattices to Quantum Kripke Frames

In this subsection, we define the functor $\mathbf{F} : \mathbb{L}^{op} \to \mathbb{F}$.[7]

3.1.1 Mapping of Objects. Fix a Piron lattice $\mathfrak{L} = (L, \leq, (\cdot)^\perp)$. We define a structure $\mathbf{F}(\mathfrak{L}) = (At(\mathfrak{L}), \to_{\mathfrak{L}})$ as follows:

1. $At(\mathfrak{L})$ is the set of all atoms of $\mathfrak{L}$;
2. $\to_{\mathfrak{L}} \stackrel{\text{def}}{=} \{(p, q) \in At(\mathfrak{L}) \times At(\mathfrak{L}) \mid p \nleq q^\perp\}$.

We will prove that $\mathbf{F}(\mathfrak{L})$ is a quantum Kripke frame by verifying the conditions in the definition one by one. (Propositions 11 to 13 below are first proved in [10]. Since the proofs are short, we present them here for the convenience of the readers.)

As a start, note that by boundedness $O \neq I$, so $At(\mathfrak{L}) \neq \emptyset$ by atomicity.

[7] By the definition of a functor, $\mathbf{F}$ is the union of two disjoint parts $\mathbf{F}_o$ and $\mathbf{F}_a$. Here $\mathbf{F}_o$ is the mapping of objects, i.e. a class function from the class of Piron lattices to the class of quantum Kripke frames; and $\mathbf{F}_a$ is the mapping of arrows, i.e. a class function from the class of $\mathbb{L}$-morphisms between Piron lattices to the class of $\mathbb{F}$-morphisms between quantum Kripke frames. However, it is customary in category theory to use the symbol $\mathbf{F}$ for both $\mathbf{F}_o$ and $\mathbf{F}_a$, which we will follow in this paper. The same notational convention also applies to $\mathbf{G}$.

Proposition 11 $\mathbf{F}(\mathfrak{L})$ satisfies reflexivity.

Proof Let $p \in At(\mathfrak{L})$ be arbitrary. Then $p \not\leq p^{\perp}$; otherwise, $p \wedge p^{\perp} = p \neq O$, contradicting the definition of orthocomplement. Therefore, $p \rightarrow_{\mathfrak{L}} p$. □

Proposition 12 $\mathbf{F}(\mathfrak{L})$ satisfies symmetry.

Proof Assume that $p, q \in At(\mathfrak{L})$ satisfies $p \rightarrow_{\mathfrak{L}} q$. By definition $p \not\leq q^{\perp}$. By the definition of orthocomplement $q = q^{\perp\perp} \not\leq p^{\perp}$. Therefore, $q \rightarrow_{\mathfrak{L}} p$. □

Proposition 13 $\mathbf{F}(\mathfrak{L})$ satisfies separation, i.e. for any two distinct $p, q \in At(\mathfrak{L})$, there is an $r \in At(\mathfrak{L})$ such that $r \rightarrow_{\mathfrak{L}} p$ and $r \not\rightarrow_{\mathfrak{L}} q$.

Proof We prove the contrapositive. Assume that $p, q \in At(\mathfrak{L})$ are such that, for each $r \in At(\mathfrak{L})$, $r \rightarrow_{\mathfrak{L}} p$ implies that $r \rightarrow_{\mathfrak{L}} q$. It follows that, for each $r \in At(\mathfrak{L})$, $r \leq q^{\perp}$ implies that $r \leq p^{\perp}$. Hence $[\![q^{\perp}]\!] \subseteq [\![p^{\perp}]\!]$. By Lemma 2 $q^{\perp} = \bigvee[\![q^{\perp}]\!] \leq \bigvee[\![p^{\perp}]\!] = p^{\perp}$. Hence $p = p^{\perp\perp} \leq q^{\perp\perp} = q$. Since $p, q \in At(\mathfrak{L})$, $p = q$. □

Proposition 14 $\mathbf{F}(\mathfrak{L})$ satisfies superposition, i.e. for any $p, q \in At(\mathfrak{L})$, there is an $r \in At(\mathfrak{L})$ such that $r \rightarrow_{\mathfrak{L}} p$ and $r \rightarrow_{\mathfrak{L}} q$.

Proof We need to consider two cases.

Case 1: $p \rightarrow_{\mathfrak{L}} q$. By Proposition 11 $p \rightarrow_{\mathfrak{L}} p$. Hence we can take $r = p$.

Case 2: $p \not\rightarrow_{\mathfrak{L}} q$. By definition $p \leq q^{\perp}$, and by Proposition 11 $p \neq q$. Then by the superposition principle there is an $r \in At(\mathfrak{L})$ such that $r \neq p$, $r \neq q$ and $r \vee p = r \vee q = p \vee q$. We show that $r \rightarrow_{\mathfrak{L}} p$ and $r \rightarrow_{\mathfrak{L}} q$.

For $r \rightarrow_{\mathfrak{L}} p$, suppose (towards a contradiction) that $r \not\rightarrow_{\mathfrak{L}} p$. By definition $r \leq p^{\perp}$. By the definition of $(\cdot)^{\perp}$ $p = p^{\perp\perp} \leq r^{\perp}$. It follows from Lemma 2 that $p \leq q^{\perp} \wedge r^{\perp} = (q \vee r)^{\perp}$. Since $q \vee r = p \vee q$, $p \leq (p \vee q)^{\perp} \leq p^{\perp}$. Hence $p \not\rightarrow_{\mathfrak{L}} p$, contradicting Proposition 11.

For $r \rightarrow_{\mathfrak{L}} q$, suppose (towards a contradiction) that $r \not\rightarrow_{\mathfrak{L}} q$, i.e. $r \leq q^{\perp}$. Then $q \leq q \vee r = p \vee r \leq q^{\perp}$; so $q \not\rightarrow_{\mathfrak{L}} q$, contradicting Proposition 11. □

Now it remains to show that $\mathbf{F}(\mathfrak{L})$ satisfies existence of approximation. Before proving this, we establish two lemmas. One of them contains two facts about the orthocomplement operation in the Kripke frame $\mathbf{F}(\mathfrak{L})$.

Lemma 15 The following hold for $\sim(\cdot) : \wp(At(\mathfrak{L})) \rightarrow \wp(At(\mathfrak{L}))$ in $\mathbf{F}(\mathfrak{L})$:

1. for each $a \in L$, $\sim[\![a]\!] = [\![a^{\perp}]\!]$;
2. for each $P \subseteq At(\mathfrak{L})$, there is an $a \in L$ such that $\sim P = [\![a]\!]$.

Proof *1* holds, because, for every $p \in At(\mathfrak{L})$,

$$p \in \sim[\![a]\!] \Leftrightarrow p \not\rightarrow_{\mathfrak{L}} q, \text{ for each } q \in [\![a]\!]$$

$$\Leftrightarrow q \leq p^{\perp}, \text{ for each atom } q \leq a$$
$$\Leftrightarrow \bigvee[\![a]\!] = \bigvee\{q \in At(\mathfrak{L}) \mid q \leq a\} \leq p^{\perp}$$
$$\Leftrightarrow a \leq p^{\perp} \qquad \text{(Lemma 2)}$$
$$\Leftrightarrow p \in [\![a^{\perp}]\!]$$

For *2*, by completeness we let $a = \bigvee \sim P$. We show that $\sim P = [\![a]\!]$.

Let $p \in At(\mathfrak{L})$ be arbitrary. First assume that $p \in \sim P$. Then $p \leq \bigvee \sim P = a$. Hence $p \in [\![a]\!]$. Second assume that $p \in [\![a]\!]$. To show that $p \in \sim P$, let $q \in P$ be arbitrary. Then $r \not\to_{\mathfrak{L}} q$ for each $r \in \sim P$. By definition $r \leq q^{\perp}$ for each $r \in \sim P$. It follows that $a = \bigvee \sim P \leq q^{\perp}$. Since $p \in [\![a]\!]$, $p \leq a$, so $p \leq q^{\perp}$, i.e. $p \not\to_{\mathfrak{L}} q$. Therefore, $p \in \sim P$. □

The other lemma shows a one-to-one correspondence between bi-orthogonally closed subsets of $At(\mathfrak{L})$ and subsets of the form $[\![a]\!]$ for some $a \in L$.

Lemma 16 For each $P \subseteq At(\mathfrak{L})$, the following are equivalent:

(i) $P = \sim\sim P$;
(ii) $P = [\![a]\!]$, for some $a \in L$.

Proof **From (i) to (ii):** By (i) and Lemma 15 $P = \sim\sim P = [\![a]\!]$ for some $a \in L$.

From (ii) to (i): Assume that $P = [\![a]\!]$. By the definition of orthocomplement $a = a^{\perp\perp}$. Then by Lemma 15 $P = [\![a]\!] = [\![a^{\perp\perp}]\!] = \sim[\![a^{\perp}]\!] = \sim\sim[\![a]\!] = \sim\sim P$. □

We are ready to tackle existence of approximation.

Proposition 17 $\mathbf{F}(\mathfrak{L})$ satisfies existence of approximation, i.e. if $p \in At(\mathfrak{L})$ and $P \subseteq At(\mathfrak{L})$ satisfy that $p \notin \sim P$ and $P = \sim\sim P$, there is a $q \in P$ such that $p \approx_P q$, i.e. $p \to_{\mathfrak{L}} r$ if and only if $q \to_{\mathfrak{L}} r$ for each $r \in P$.

Proof Assume that $p \in At(\mathfrak{L})$ and $P \subseteq At(\mathfrak{L})$ satisfy that $p \notin \sim P$ and $P = \sim\sim P$. By Lemma 16 $P = [\![a]\!]$ for some $a \in L$. By Lemma 15 $\sim P = [\![a^{\perp}]\!]$. Since $p \notin \sim P$, $p \not\leq a^{\perp}$, so $p \wedge a^{\perp} \neq p$. Since $p \in At(\mathfrak{L})$, $p \wedge a^{\perp} = O$. By the covering law $q \overset{\text{def}}{=} (p \vee a^{\perp}) \wedge a = (p \vee a^{\perp}) \wedge a^{\perp\perp} \in At(\mathfrak{L})$. We show q has the required property.

First show that $q \in P$. Since $q = (p \vee a^{\perp}) \wedge a$, $q \leq a$, so $q \in [\![a]\!] = P$.

Second show that $p \to_{\mathfrak{L}} r$ if and only if $q \to_{\mathfrak{L}} r$ for each $r \in P$. Let $r \in P$ be arbitrary. Since $r \in P = [\![a]\!]$, $r \leq a$, so $a^{\perp} \leq r^{\perp}$ by the definition of $(\cdot)^{\perp}$. Moreover, since $a^{\perp} \leq p \vee a^{\perp}$, by Lemma 2 $p \vee a^{\perp} = a^{\perp} \vee ((p \vee a^{\perp}) \wedge a^{\perp\perp})$, so by the definition of $(\cdot)^{\perp}$ $p \vee a^{\perp} = a^{\perp} \vee ((p \vee a^{\perp}) \wedge a) = a^{\perp} \vee q$. Therefore, we have: $p \not\to_{\mathfrak{L}} r \;\Leftrightarrow\; p \leq r^{\perp} \;\Leftrightarrow\; p \vee a^{\perp} \leq r^{\perp} \;\Leftrightarrow\; a^{\perp} \vee q \leq r^{\perp} \;\Leftrightarrow\; q \leq r^{\perp} \;\Leftrightarrow\; q \not\to_{\mathfrak{L}} r$ □

Theorem 18 $\mathbf{F}(\mathfrak{L}) = (At(\mathfrak{L}), \to_{\mathfrak{L}})$ is a quantum Kripke frame. Moreover, $\mathbf{F}$ is a class function from the class of Piron lattices to the class of quantum Kripke frames.

Proof It follows from Propositions 11, 12, 13, 14 and 17. □

3.1.2 Mapping of Arrows. Fix an $\mathbb{L}$-morphism h from $\mathfrak{L}_1 = (L_1, \leq_1, (\cdot)^{\perp_1})$, a Piron lattice, to $\mathfrak{L}_2 = (L_2, \leq_2, (\cdot)^{\perp_2})$. By definition h preserves meets, so it has a left adjoint:

$$\ell_h : L_2 \to L_1 :: a_2 \mapsto \bigwedge\{a_1 \in L_1 \mid a_2 \leq_2 h(a_1)\}$$

Hence ℓ_h satisfies that, for any $a_1 \in L_1$ and $a_2 \in L_2$, $\ell_h(a_2) \leq_1 a_1 \Leftrightarrow a_2 \leq_2 h(a_1)$. Moreover, we can prove the following:

Lemma 19 ℓ_h maps each element in $At(\mathfrak{L}_2)$ to either an element in $At(\mathfrak{L}_1)$ or O_1.

Proof It follows directly from Moore's condition in the definition of $\mathbb{L}$-morphisms. A proof can be found in the literature, e.g. the proof of Lemma 4.3 in [10]. □

Now we define a partial function $\mathbf{F}(h) : At(\mathfrak{L}_2) \dashrightarrow At(\mathfrak{L}_1)$ as follows:

$$\mathbf{F}(h)(p_2) = \begin{cases} \ell_h(p_2), & \text{if } \ell_h(p_2) \in At(\mathfrak{L}_1) \\ \text{undefined}, & \text{if } \ell_h(p_2) = O_1 \end{cases}$$

We continue to show that $\mathbf{F}(h)$ is well defined.

Theorem 20 $\mathbf{F}(h)$ is an $\mathbb{F}$-morphism from $\mathbf{F}(\mathfrak{L}_2)$ to $\mathbf{F}(\mathfrak{L}_1)$. Moreover, $\mathbf{F}$ is a class function from the class of $\mathbb{L}$-morphisms to the class of $\mathbb{F}$-morphisms.

Proof Since ℓ_h is a function, by definition $\mathbf{F}(h)$ is a partial function. It remains to show that $\mathbf{F}(h)$ satisfies the defining condition of $\mathbb{F}$-morphisms.

Assume that $p_2 \in At(\mathfrak{L}_2)$ and $p_1, q_1 \in At(\mathfrak{L}_1)$ satisfy that $\mathbf{F}(h)(p_2) = p_1$ and $p_1 \to_{\mathfrak{L}_1} q_1$. By Lemma 2 $q_1^{\perp_1} \in coAt(\mathfrak{L}_1)$. By dual Moore's condition there is a $b_2 \in coAt(\mathfrak{L}_2)$ such that $b_2 \leq_2 h(q_1^{\perp_1})$. By Lemma 2 $q_2 \overset{\text{def}}{=} b_2^{\perp_2} \in At(\mathfrak{L}_2)$.

I claim that $h(q_1^{\perp_1}) = q_2^{\perp_2}$. By the above $(h(q_1^{\perp_1}))^{\perp_2} \leq_2 b_2^{\perp_2} = q_2$. Since $q_2 \in At(\mathfrak{L}_2)$, $(h(q_1^{\perp_1}))^{\perp_2} = O_2$ or $(h(q_1^{\perp_1}))^{\perp_2} = q_2$. By the assumption $\ell_h(p_2) = \mathbf{F}(h)(p_2) = p_1 \not\leq_1 q_1^{\perp_1}$. By the definition of ℓ_h $p_2 \not\leq_2 h(q_1^{\perp_1})$, so $(h(q_1^{\perp_1}))^{\perp_2} \not\leq_2 p_2^{\perp_2}$. Hence $(h(q_1^{\perp_1}))^{\perp_2} \neq O_2$. Therefore, $(h(q_1^{\perp_1}))^{\perp_2} = q_2$, so $h(q_1^{\perp_1}) = q_2^{\perp_2}$.

Now we show that (q_1, q_2) satisfies (Ad)$_{\mathbf{F}(h)}$. Let $r_2 \in At(\mathfrak{L}_2)$ be arbitrary.

$$\begin{aligned} q_2 \to_{\mathfrak{L}_2} r_2 &\Leftrightarrow r_2 \not\leq_2 q_2^{\perp_2} \\ &\Leftrightarrow r_2 \not\leq_2 h(q_1^{\perp_1}) && \text{(the claim)} \\ &\Leftrightarrow \ell_h(r_2) \not\leq_1 q_1^{\perp_1} \\ &\Leftrightarrow \ell_h(r_2) \neq O_1 \text{ and } \ell_h(r_2) \to_{\mathfrak{L}_1} q_1 \\ &\Leftrightarrow \mathbf{F}(h)(r_2) \text{ is defined and } q_1 \to_{\mathfrak{L}_1} \mathbf{F}(h)(r_2) \end{aligned}$$

Therefore, $\mathbf{F}(h)$ is an $\mathbb{F}$-morphism. □

Theorem 21 **F** is a functor from $\mathbb{L}^{op}$ to $\mathbb{F}$.

Proof By Theorems 18 and 20 **F** maps objects in $\mathbb{L}$ to objects in $\mathbb{F}$, and arrows $h : \mathfrak{L}_1 \to \mathfrak{L}_2$ in $\mathbb{L}$ to arrows $\mathbf{F}(h) : \mathbf{F}(\mathfrak{L}_2) \to \mathbf{F}(\mathfrak{L}_1)$ in $\mathbb{F}$.

We show that **F** preserves identity arrows. For each $id_{\mathfrak{L}} : \mathfrak{L} \to \mathfrak{L}$, by definition

$$\ell_{id_{\mathfrak{L}}} :: a \mapsto \bigwedge\{b \in L \mid a \leq id_{\mathfrak{L}}(b)\} = \bigwedge\{b \in L \mid a \leq b\} = a$$

It follows easily that $\mathbf{F}(id_{\mathfrak{L}}) = Id_{At(\mathfrak{L})} = id_{\mathbf{F}(\mathfrak{L})}$.

We show that **F** preserves composition. Let $h : \mathfrak{L}_1 \to \mathfrak{L}_2$ and $g : \mathfrak{L}_2 \to \mathfrak{L}_3$ be two $\mathbb{L}$-morphisms. Then, for each $(p_3, p_1) \in At(\mathfrak{L}_3) \times At(\mathfrak{L}_1)$,

$$\begin{aligned}
(p_3, p_1) \in \mathbf{F}(h) \circ \mathbf{F}(g) &\Leftrightarrow (p_3, p_2) \in \mathbf{F}(g) \text{ and } (p_2, p_1) \in \mathbf{F}(h), \text{ for a } p_2 \in At(\mathfrak{L}_2) \\
&\Leftrightarrow \ell_g(p_3) = p_2 \text{ and } \ell_h(p_2) = p_1, \text{ for some } p_2 \in At(\mathfrak{L}_2) \\
&\Leftrightarrow \ell_h(\ell_g(p_3)) = p_1 \qquad\qquad (\text{note that } \ell_h(O_2) = O_1) \\
&\Leftrightarrow p_1 = \bigwedge\{a_1 \in \Sigma_1 \mid \ell_g(p_3) \leq_2 h(a_1)\} \\
&\Leftrightarrow p_1 = \bigwedge\{a_1 \in \Sigma_1 \mid p_3 \leq_3 (g \circ h)(a_1)\} \\
&\Leftrightarrow p_1 = \ell_{g \circ h}(p_3) \\
&\Leftrightarrow (p_3, p_1) \in \mathbf{F}(g \circ h)
\end{aligned}$$

Therefore, $\mathbf{F}(h) \circ \mathbf{F}(g) = \mathbf{F}(g \circ h)$. □

3.2 From Quantum Kripke Frames to Piron Lattices

In this subsection, we define the functor $\mathbf{G} : \mathbb{F}^{op} \to \mathbb{L}$.

3.2.1 Mapping of Objects. Fix a quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$. Define $\mathbf{G}(\mathfrak{F})$ to be the tuple $(\mathcal{L}_{\mathfrak{F}}, \subseteq, \sim(\cdot))$. Recall that $\sim P = \{s \in \Sigma \mid s \not\to t$, for each $t \in P\}$ for each $P \subseteq \Sigma$, and $\mathcal{L}_{\mathfrak{F}}$ is the set of all $P \subseteq \Sigma$ satisfying $P = \sim\sim P$. We show that $\mathbf{G}(\mathfrak{F})$ is a Piron lattice by verifying the conditions in the definition one by one. (Propositions 22 to 25 below are first proved in [3] and Proposition 27 in [10]. Since the proofs are short, we present them here for the convenience of the readers.)

Proposition 22 In $\mathbf{G}(\mathfrak{F})$, $(\mathcal{L}_{\mathfrak{F}}, \subseteq)$ is a lattice.

Proof It is well-known that $\subseteq$ is a partial order. For any $P, Q \in \mathcal{L}_{\mathfrak{F}}$, it is not hard to show that $P \cap Q \in \mathcal{L}_{\mathfrak{F}}$ by Lemma 8 and it is the infimum of P and Q. Moreover, by Lemma 8 $\sim(\sim P \cap \sim Q) \in \mathcal{L}_{\mathfrak{F}}$. We show that it is the supremum of P and Q.

Since $\sim P \cap \sim Q \subseteq \sim P$, by Lemma 7 $P \subseteq \sim\sim P \subseteq \sim(\sim P \cap \sim Q)$. Similar result holds for Q, so $\sim(\sim P \cap \sim Q)$ is an upper bound of P and Q.

Now assume that $R \in \mathcal{L}_{\mathfrak{F}}$ satisfies $P \subseteq R$ and $Q \subseteq R$. By Lemma 7 $\sim R \subseteq \sim P$ and $\sim R \subseteq \sim Q$. Hence $\sim R \subseteq \sim P \cap \sim Q$, and thus $\sim(\sim P \cap \sim Q) \subseteq \sim\sim R = R$. Therefore, $\sim(\sim P \cap \sim Q)$ is the supremum of P and Q in $(\mathcal{L}_{\mathfrak{F}}, \subseteq)$. □

We use $P \sqcup Q$ to denote $\sim(\sim P \cap \sim Q)$, and we generalize this notation:

$$\bigsqcup_{i \in I} P_i \overset{\text{def}}{=} \sim \bigcap_{i \in I} \sim P_i, \text{ for any } \{P_i \in \mathcal{L}_{\mathfrak{F}} \mid i \in I\}$$

Note that by Lemma 8 $\bigsqcup_{i \in I} P_i \in \mathcal{L}_{\mathfrak{F}}$, and by definition $\bigsqcup_{i \in I} P_i = \sim\sim \bigcup_{i \in I} P_i$, where $\bigcup$ is the set-theoretic union. Moreover, we can prove the following:

Proposition 23 $\mathbf{G}(\mathfrak{F})$ satisfies completeness: for any $\{P_i \in \mathcal{L}_{\mathfrak{F}} \mid i \in I\}$, $\bigcap_{i \in I} P_i \in \mathcal{L}_{\mathfrak{F}}$ is the infimum and $\bigsqcup_{i \in I} P_i \in \mathcal{L}_{\mathfrak{F}}$ is the supremum.

Proof By Lemma 8 $\bigcap_{i \in I} P_i, \bigsqcup_{i \in I} P_i \in \mathcal{L}_{\mathfrak{F}}$. A generalization of the proof of Proposition 22 shows that $\bigcap_{i \in I} P_i$ is the infimum and $\bigsqcup_{i \in I} P_i$ is the supremum. □

Proposition 24 $\mathbf{G}(\mathfrak{F})$ satisfies boundedness, i.e. $\emptyset, \Sigma \in \mathcal{L}_{\mathfrak{F}}$, $\emptyset \neq \Sigma$ and, for each $P \in \mathcal{L}_{\mathfrak{F}}$, $\emptyset \subseteq P \subseteq \Sigma$.

Proof Use Lemma 7 and the definitions of quantum Kripke frames and of $\mathcal{L}_{\mathfrak{F}}$ □

Proposition 25 $\mathbf{G}(\mathfrak{F})$ is orthocomplemented, where $\sim(\cdot)$ is an orthocomplementation, i.e. it is a function on $\mathcal{L}_{\mathfrak{F}}$ satisfying: for any $P, Q \in \mathcal{L}_{\mathfrak{F}}$,

1. $P \cap \sim P = \emptyset$ and $P \sqcup \sim P = \Sigma$;
2. $P \subseteq Q$ implies that $\sim Q \subseteq \sim P$;
3. $\sim\sim P = P$.

Proof When restricted to $\mathcal{L}_{\mathfrak{F}}$, by Lemma 8 $\sim(\cdot) : \wp(\Sigma) \to \wp(\Sigma)$ is a function on $\mathcal{L}_{\mathfrak{F}}$. Note that by definition and Lemma 7 $P \sqcup \sim P = \sim(\sim P \cap \sim\sim P) = \sim(\sim P \cap P) = \sim\emptyset = \Sigma$. The other items are contained in Lemma 7. □

Before continuing to tackle atomicity, we prove a useful lemma characterizing the atoms and coatoms of $\mathbf{G}(\mathfrak{F})$.

Lemma 26 In $\mathbf{G}(\mathfrak{F})$, the following hold:

1. $\{s\} \in \mathcal{L}_{\mathfrak{F}}$, for each $s \in \Sigma$;
2. for each $P \in \mathcal{L}_{\mathfrak{F}}$, $P \in At(\mathbf{G}(\mathfrak{F})) \Leftrightarrow P = \{s\}$ for some $s \in \Sigma$;
3. for each $P \in \mathcal{L}_{\mathfrak{F}}$, $P \in coAt(\mathbf{G}(\mathfrak{F})) \Leftrightarrow P = \sim\{s\}$ for some $s \in \Sigma$.

Proof For *1*, let $s \in \Sigma$ be arbitrary. By Lemma 7 $\{s\} \subseteq \sim\sim\{s\}$. For the other direction, assume that $t \notin \{s\}$. Then $t \neq s$. By separation there is a $w \in \Sigma$ such that $w \to t$ and $w \not\to s$. For $w \not\to s$, $w \in \sim\{s\}$. Since $w \to t$, by symmetry $t \to w$ and thus $t \notin \sim\sim\{s\}$. Therefore, $\{s\} = \sim\sim\{s\}$, and thus $\{s\} \in \mathcal{L}_{\mathfrak{F}}$.

For *2*, the '$\Leftarrow$' direction is obvious. For the '$\Rightarrow$' direction, assume that $P \in At(\mathbf{G}(\mathfrak{F}))$. By definition $P \neq \emptyset$, so there is an $s \in P$. Then $\emptyset \subseteq \{s\} \subseteq P$ and $\{s\} \in \mathcal{L}_{\mathfrak{F}}$ by *1*. Since $P \in At(\mathbf{G}(\mathfrak{F}))$ and $\{s\} \neq \emptyset$, $P = \{s\}$.

For *3*, note that the proof of Item *2* in Lemma 2 only uses boundedness and orthocomplementation, so by Propositions 24 and 25 we can use this result here:

$$\begin{aligned} P \in coAt(\mathbf{G}(\mathfrak{F})) &\Leftrightarrow {\sim}P \in At(\mathbf{G}(\mathfrak{F})) && \text{(by Item } \mathit{2} \text{ in Lemma 2)} \\ &\Leftrightarrow {\sim}P = \{s\}, \text{ for some } s \in \Sigma && \text{(by } \mathit{2}\text{)} \\ &\Leftrightarrow P = {\sim}\{s\}, \text{ for some } s \in \Sigma && \square \end{aligned}$$

Proposition 27 $\mathbf{G}(\mathfrak{F})$ satisfies atomicity.

Proof Assume that $P \in \mathcal{L}_{\mathfrak{F}}$ satisfies $P \neq \emptyset$. Then $s \in P$ for some $s \in \Sigma$. By the previous lemma $\{s\} \in At(\mathbf{G}(\mathfrak{F}))$ and $\{s\} \subseteq P$. □

Proposition 28 $\mathbf{G}(\mathfrak{F})$ satisfies weak modularity, i.e. for any $P, Q \in \mathcal{L}_{\mathfrak{F}}$, $P \subseteq Q$ implies that $P = Q \cap (P \sqcup {\sim}Q)$.

Proof Assume that $P \subseteq Q$. By this assumption and the property of $\sqcup$, $P \subseteq Q$ and $P \subseteq P \sqcup {\sim}Q$, and thus $P \subseteq Q \cap (P \sqcup {\sim}Q)$. It remains to show that $Q \cap ({\sim}Q \sqcup P) \subseteq P$.

Let $s \in Q \cap ({\sim}Q \sqcup P)$ be arbitrary. Then $s \in Q$ and $s \in {\sim}Q \sqcup P = {\sim}(Q \cap {\sim}P)$. Since $P \in \mathcal{L}_{\mathfrak{F}}$, to show that $s \in P$, it suffices to show that $s \in {\sim\sim}P$. Hence we let $t \in {\sim}P$ be arbitrary and try to show that $s \not\to t$. Consider two cases.

Case 1: $t \in {\sim}Q$. Since $s \in Q$, it follows that $s \not\to t$.

Case 2: $t \notin {\sim}Q$. Since $Q \in \mathcal{L}_{\mathfrak{F}}$, by existence of approximation there is a $t_{\|} \in Q$ such that $t \approx_Q t_{\|}$. Since $P \subseteq Q$, $t \approx_P t_{\|}$. For $t \in {\sim}P$, $t_{\|} \in {\sim}P$. Hence $t_{\|} \in Q \cap {\sim}P$. Since $s \in {\sim}(Q \cap {\sim}P)$, $t_{\|} \not\to s$. Since $s \in Q$, by the definition of $t_{\|}$, $t \not\to s$, so $s \not\to t$. □

Proposition 29 $\mathbf{G}(\mathfrak{F})$ satisfies the covering law, i.e. for any $s \in \Sigma$ and $P \in \mathcal{L}_{\mathfrak{F}}$ such that $\{s\} \cap P = \emptyset$, $(P \sqcup \{s\}) \cap {\sim}P \in At(\mathbf{G}(\mathfrak{F}))$.

Proof Assume that $\{s\} \cap P = \emptyset$. Then $s \notin P = {\sim\sim}P$. By Lemma 8 ${\sim}P \in \mathcal{L}_{\mathfrak{F}}$. Hence by existence of approximation $s \approx_{\sim P} s_{\perp}$ for some $s_{\perp} \in {\sim}P$. I claim that $(P \sqcup \{s\}) \cap {\sim}P = \{s_{\perp}\}$, implying $(P \sqcup \{s\}) \cap {\sim}P \in At(\mathbf{G}(\mathfrak{F}))$ by Lemma 26.

First show that $\{s_{\perp}\} \subseteq (P \sqcup \{s\}) \cap {\sim}P$. By definition $s_{\perp} \in {\sim}P$. To show that $s_{\perp} \in P \sqcup \{s\} = {\sim}({\sim}P \cap {\sim}\{s\})$, let $t \in {\sim}P \cap {\sim}\{s\}$ be arbitrary. Since $t \in {\sim}\{s\}$, $s \not\to t$. Since $t \in {\sim}P$, $s_{\perp} \not\to t$ by the definition of $s_{\perp}$. As a result, $s_{\perp} \in (P \sqcup \{s\}) \cap {\sim}P$.

Second show that $(P \sqcup \{s\}) \cap {\sim}P \subseteq \{s_{\perp}\}$. Suppose (towards a contradiction) that there is an $s' \in (P \sqcup \{s\}) \cap {\sim}P$ such that $s' \neq s_{\perp}$. Then by separation there is a $t \in \Sigma$ such that $s_{\perp} \not\to t$ and $s' \to t$. Since $s' \in {\sim}P$ and $s' \to t$, $t \notin {\sim\sim}P$. By existence of approximation there is a $t_{\perp} \in {\sim}P$ such that $t \approx_{\sim P} t_{\perp}$. On the one hand, $s_{\perp}, s' \in {\sim}P$, we get $s_{\perp} \not\to t_{\perp}$ and $s' \to t_{\perp}$. On the other hand, it follows from $t_{\perp} \in {\sim}P$, $s \approx_{\sim P} s_{\perp}$ and $s_{\perp} \not\to t_{\perp}$ that $s \not\to t_{\perp}$. Hence $t_{\perp} \in {\sim}P \cap {\sim}\{s\}$.

Since $s' \in P \sqcup \{s\} = \sim(\sim P \cap \sim\{s\})$, $s' \not\to t_\perp$, which contradicts what we have got before. As a result, $(P \sqcup \{s\}) \cap \sim P \subseteq \{s_\perp\}$. □

Now it remains to tackle the superposition principle. Before doing this, we prove a lemma. In the following, for convenience we write $s \sqcup t$ for $\{s\} \sqcup \{t\} = \sim\sim\{s, t\}$.

Lemma 30 For any $s, t, w \in \Sigma$ such that $w \neq t$ and $w \in s \sqcup t$, $s \sqcup t = w \sqcup t$.

Proof First, note that $s \neq t$; otherwise, $w \in s \sqcup t = \{t\}$, contradicting $w \neq t$.

Second, since $s \neq t$, $s \notin \{t\} = \sim\sim\{t\}$ by Lemma 26, so by existence of approximation there is an $s' \in \sim\{t\}$ such that $s \approx_{\sim\{t\}} s'$. Similarly, since $w \neq t$, $w \notin \{t\} = \sim\sim\{t\}$, so there is a $w' \in \sim\{t\}$ such that $w \approx_{\sim\{t\}} w'$.

I claim that $w' = s'$. Suppose (towards a contradiction) that $w' \neq s'$. By separation there is a $v \in \Sigma$ such that $v \to w'$ and $v \not\to s'$. Since $v \to w'$ and $w' \in \sim\{t\}$, $v \notin \sim\sim\{t\}$. By existence of approximation there is a $v' \in \sim\{t\}$ such that $v \approx_{\sim\{t\}} v'$. Now, on the one hand, since $w', s' \in \sim\{t\}$ and $v \approx_{\sim\{t\}} v'$, we get that $v' \to w'$ and $v' \not\to s'$. On the other hand, since $v' \to w'$ and $v' \in \sim\{t\}$, it follows from $w \approx_{\sim\{t\}} w'$ that $v' \to w$. Since $w \in s \sqcup t = \sim\sim\{s, t\}$, $v' \notin \sim\{s, t\}$, i.e. $v' \to s$ or $v' \to t$. Since $v' \in \sim\{t\}$, $v' \to s$ holds. Since again $v' \in \sim\{t\}$, it follows from $s \approx_{\sim\{t\}} s'$ that $v' \to s'$, contradicting $v' \not\to s'$ which we have got before. Therefore, $w' = s'$.

Since $s \approx_{\sim\{t\}} s'$ and $w \approx_{\sim\{t\}} w'$ mean the same as $\sim\{s, t\} = \sim\{s', t\}$ and $\sim\{w, t\} = \sim\{w', t\}$, respectively, $s \sqcup t = \sim\sim\{s, t\} = \sim\sim\{s', t\} = \sim\sim\{w', t\} = \sim\sim\{w, t\} = w \sqcup t$. □

Proposition 31 $\mathbf{G}(\mathfrak{F})$ satisfies the superposition principle, i.e. for any $s, t \in \Sigma$ satisfying $s \neq t$, there is an $r \in \Sigma \setminus \{s, t\}$ such that $s \sqcup t = r \sqcup s = r \sqcup t$.

Proof We consider two cases.

Case 1: $s \to t$. Since $s \neq t$, by separation there is an $r' \in \Sigma$ such that $r' \not\to s$ and $r' \to t$. Since $r' \to t$, $r' \notin \sim\{s, t\} = \sim(s \sqcup t)$. By existence of approximation there is an $r \in s \sqcup t$ such that $r \approx_{s \sqcup t} r'$. Since $s, t \in s \sqcup t$, $r \not\to s$ and $r \to t$. Since $r \not\to s$, by reflexivity $r \neq s$. Since $r \not\to s$ and $s \to t$, $r \neq t$. From $r \neq s$, $s \neq t$, $r \neq t$ and $r \in s \sqcup t$, it is easy to see from Lemma 30 that $s \sqcup t = r \sqcup s = r \sqcup t$.

Case 2: $s \not\to t$. By superposition there is an $r' \in \Sigma$ such that $r' \to s$ and $r' \to t$. Since $r' \to t$, $r' \notin \sim\{s, t\} = \sim(s \sqcup t)$. By existence of approximation there is an $r \in s \sqcup t$ such that $r \approx_{s \sqcup t} r'$. Since $s, t \in s \sqcup t$, $r \to s$ and $r \to t$. Since $s \not\to t$ and $r \to t$, $r \neq s$. Similarly, we know that $r \neq t$. From $r \neq s$, $s \neq t$, $r \neq t$ and $r \in s \sqcup t$, it is easy to see from Lemma 30 that $s \sqcup t = r \sqcup s = r \sqcup t$. □

Theorem 32 $\mathbf{G}(\mathfrak{F}) = (\mathcal{L}_{\mathfrak{F}}, \subseteq, \sim(\cdot))$ is a Piron lattice. Moreover, $\mathbf{G}$ is a class function from the class of quantum Kripke frames to the class of Piron lattices.

Proof It follows from Propositions 22, 23, 24, 25, 27, 28, 29 and 31. □

3.2.2 Mapping of Arrows. Fix an $\mathbb{F}$-morphism $F : \Sigma_1 \dashrightarrow \Sigma_2$ from a quantum Kripke frame $\mathfrak{F}_1 = (\Sigma_1, \to_1)$ to $\mathfrak{F}_2 = (\Sigma_2, \to_2)$. We denote by $\text{Ker}(F)$ the set $\{s_1 \in \Sigma_1 \mid F(s_1) \text{ is undefined}\}$. To define the functor **F**, we need a lemma.

Lemma 33 For each $P_2 \in \mathcal{L}_{\mathfrak{F}_2}$, $\text{Ker}(F) \cup F^{-1}[P_2] \in \mathcal{L}_{\mathfrak{F}_1}$, where $F^{-1}[P_2] = \{s_1 \in \Sigma_1 \mid F(s_1) = s_2 \text{ for some } s_2 \in P_2\}$.

Proof Assume that $P_2 \in \mathcal{L}_{\mathfrak{F}_2}$. By Lemma 7 $\text{Ker}(F) \cup F^{-1}[P_2] \subseteq {\sim}{\sim}(\text{Ker}(F) \cup F^{-1}[P_2])$. For the other direction, let $w_1 \notin \text{Ker}(F) \cup F^{-1}[P_2]$ be arbitrary. Then $F(w_1)$ is defined and $F(w_1) \notin P_2$. For $P_2 \in \mathcal{L}_{\mathfrak{F}_2}$, $P_2 = {\sim}{\sim}P_2$, so $F(w_1) \to_2 t_2$ for some $t_2 \in {\sim}P_2$. By the definition of $\mathbb{F}$-morphisms there is a $t_1 \in \Sigma_1$ such that (t_1, t_2) satisfies the following :

$$(\text{Ad})_F \quad \text{for every } s_1 \in \Sigma_1, t_1 \to_1 s_1 \iff \Big(F(s_1) \text{ is defined and } t_2 \to_2 F(s_1)\Big)$$

Since $F(w_1)$ is defined and $F(w_1) \to_2 t_2$, by symmetry and $(\text{Ad})_F$ $w_1 \to_1 t_1$.

I claim that $t_1 \in {\sim}(\text{Ker}(F) \cup F^{-1}[P_2])$. To show this, let $r_1 \in \text{Ker}(F) \cup F^{-1}[P_2]$ be arbitrary. If $r_1 \in \text{Ker}(F)$, by $(\text{Ad})_F$ $t_1 \not\to_1 r_1$. If $r_1 \in F^{-1}[P_2]$, $F(r_1)$ is defined and $F(r_1) \in P_2$. Since $t_2 \in {\sim}P_2$, $t_2 \not\to_2 F(r_1)$. Hence by $(\text{Ad})_F$ $t_1 \not\to_1 r_1$.

Combining this claim with $w_1 \to_1 t_1$, $w_1 \notin {\sim}{\sim}(\text{Ker}(F) \cup F^{-1}[P_2])$. □

Using this lemma, we define the following function:

$$\mathbf{G}(F) : \mathcal{L}_{\mathfrak{F}_2} \to \mathcal{L}_{\mathfrak{F}_1} :: P_2 \mapsto \text{Ker}(F) \cup F^{-1}[P_2]$$

We show that $\mathbf{G}(F)$ is well defined.

Theorem 34 $\mathbf{G}(F)$ is an $\mathbb{L}$-morphism from $\mathbf{G}(\mathfrak{F}_2)$ to $\mathbf{G}(\mathfrak{F}_1)$. Moreover, **G** is a class function from the class of $\mathbb{F}$-morphisms to the class of $\mathbb{L}$-morphisms.

Proof First we verify meet preservation[8]. For any $\{P_2^i \in \mathcal{L}_{\mathfrak{F}_2} \mid i \in I\}$, if $I \neq \emptyset$,

$$\begin{aligned}\mathbf{G}(F)(\bigcap_{i\in I} P_2^i) = \text{Ker}(F) \cup F^{-1}[\bigcap_{i\in I} P_2^i] &= \text{Ker}(F) \cup \bigcap_{i\in I} F^{-1}[P_2^i] \\ &= \bigcap_{i\in I} \big(\text{Ker}(F) \cup F^{-1}[P_2^i]\big) \\ &= \bigcap_{i\in I} \mathbf{G}(F)(P_2^i);\end{aligned}$$

if $I = \emptyset$, $\mathbf{G}(F)(\bigcap_{i\in I} P_2^i) = \mathbf{G}(F)(\Sigma_2) = \text{Ker}(F) \cup F^{-1}[\Sigma_2] = \Sigma_1 = \bigcap_{i\in I} \mathbf{G}(F)(P_2^i)$.

[8] Since subscripts are used to distinguish between objects from the two quantum Kripke frames, in this proof we use superscripts as indices.

Second we verify Moore's condition. Assume that $P_1 \in At(\mathbf{G}(\mathfrak{F}_1))$. By Lemma 26 $P_1 = \{s_1\}$ for some $s_1 \in \Sigma_1$. Consider two cases.

Case 1: $s_1 \in \mathrm{Ker}(F)$. For $\Sigma_2 \neq \emptyset$, there is an $s_2 \in \Sigma_2$. By Lemma 26 $\{s_2\} \in At(\mathbf{G}(\mathfrak{F}_2))$. Then $P_1 = \{s_1\} \subseteq \mathrm{Ker}(F) \subseteq \mathrm{Ker}(F) \cup F^{-1}[\{s_2\}] = \mathbf{G}(F)(\{s_2\})$.

Case 2: $s_1 \notin \mathrm{Ker}(F)$. Then $P_1 = \{s_1\} \subseteq \mathrm{Ker}(F) \cup F^{-1}[\{F(s_1)\}] = \mathbf{G}(F)(\{F(s_1)\})$ and by Lemma 26 $\{F(s_1)\} \in At(\mathbf{G}(\mathfrak{F}_2))$.

Third we verify the dual Moore's condition. Assume that $P_2 \in coAt(\mathbf{G}(\mathfrak{F}_2))$. By Lemma 26 $P_2 = \sim\{t_2\}$ for some $t_2 \in \Sigma_2$. Then $\mathbf{G}(F)(P_2) = \mathbf{G}(F)(\sim\{t_2\}) = \mathrm{Ker}(F) \cup F^{-1}[\sim\{t_2\}]$. Consider two cases.

Case 1: $\mathbf{G}(F)(P_2) = \Sigma_1$. Since $\Sigma_1 \neq \emptyset$, pick $t_1 \in \Sigma_1$. Then $\sim\{t_1\} \in coAt(\mathbf{G}(\mathfrak{F}_1))$ by Lemma 26. Obviously $\sim\{t_1\} \subseteq \Sigma_1 = \mathbf{G}(F)(P_2)$.

Case 2: $\mathrm{Ker}(F) \cup F^{-1}[\sim\{t_2\}] = \mathbf{G}(F)(P_2) \neq \Sigma_1$. Then there is a $w_1 \in \Sigma_1$ such that $w_1 \notin \mathrm{Ker}(F) \cup F^{-1}[\sim\{t_2\}]$. Hence $F(w_1)$ is defined and $F(w_1) \rightarrow_2 t_2$. By the definition of $\mathbb{F}$-morphisms there is a $t_1 \in \Sigma_1$ such that, for every $s_1 \in \Sigma_1$, $F(s_1)$ is defined and $t_2 \rightarrow_2 F(s_1)$, if and only if $t_1 \rightarrow_1 s_1$. This means that $\mathrm{Ker}(F) \cup F^{-1}[\sim\{t_2\}] = \sim\{t_1\}$. Therefore, $\sim\{t_1\} \in coAt(\mathbf{G}(\mathfrak{F}_1))$ is such that $\sim\{t_1\} \subseteq \mathrm{Ker}(F) \cup F^{-1}[\sim\{t_2\}] = \mathbf{G}(F)(P_2)$. □

Theorem 35 **G** is a functor from $\mathbb{F}^{op}$ to $\mathbb{L}$.

Proof By Theorems 32 and 34 **G** maps objects in $\mathbb{F}$ to objects in $\mathbb{L}$, and maps arrows $F : \mathfrak{F}_1 \to \mathfrak{F}_2$ in $\mathbb{F}$ to arrows $\mathbf{G}(F) : \mathbf{G}(\mathfrak{F}_2) \to \mathbf{G}(\mathfrak{F}_1)$ in $\mathbb{L}$.

We show that **G** preserves identity arrows. For each $id_{\mathfrak{F}} : \mathfrak{F} \to \mathfrak{F}$,

$$\mathbf{G}(id_{\mathfrak{F}}) :: P \mapsto \mathrm{Ker}(id_{\mathfrak{F}}) \cup id_{\mathfrak{F}}^{-1}[P] = \mathrm{Ker}(Id_{\Sigma}) \cup Id_{\Sigma}^{-1}[P] = \emptyset \cup P = P$$

It follows that $\mathbf{G}(id_{\mathfrak{F}}) = id_{\mathbf{G}(\mathfrak{F})}$.

We show that **G** preserves composition. Let $F : \mathfrak{F}_1 \to \mathfrak{F}_2$ and $G : \mathfrak{F}_2 \to \mathfrak{F}_3$ be two $\mathbb{F}$-morphisms. Then, $\mathbf{G}(G \circ F) = \mathbf{G}(F) \circ \mathbf{G}(G)$, because, for any $P_3 \in \mathcal{L}_{\mathfrak{F}_3}$,

$$\begin{aligned}
\big(\mathbf{G}(F) \circ \mathbf{G}(G)\big)(P_3) &= \mathbf{G}(F)(\mathrm{Ker}(G) \cup G^{-1}[P_3]) \\
&= \mathrm{Ker}(F) \cup F^{-1}[\mathrm{Ker}(G) \cup G^{-1}[P_3]] \\
&= \mathrm{Ker}(F) \cup F^{-1}[\mathrm{Ker}(G)] \cup (F^{-1} \circ G^{-1})[P_3] \\
&= \mathrm{Ker}(G \circ F) \cup (G \circ F)^{-1}[P_3] \\
&= \mathbf{G}(G \circ F)(P_3)
\end{aligned}$$ □

4 The Duality

In this section we define natural isomorphisms $\tau : 1_{\mathbb{L}} \to \mathbf{G} \circ \mathbf{F}$ and $\eta : 1_{\mathbb{F}} \to \mathbf{F} \circ \mathbf{G}$, and show that $(\mathbf{F}, \mathbf{G}, \tau, \eta)$ forms a duality between the categories $\mathbb{L}$ and $\mathbb{F}$.

First we define τ. For each Piron lattice $\mathfrak{L} = (L, \leq, (\cdot)^{\perp})$, define a function

$$\tau_{\mathfrak{L}} : L \to \mathcal{L}_{\mathbf{F}(\mathfrak{L})} :: a \mapsto [\![a]\!]$$

Lemma 36 For each Piron lattice $\mathfrak{L} = (L, \leq, (\cdot)^{\perp})$, $\tau_{\mathfrak{L}}$ is an isomorphism in $\mathbb{L}$ from $1_{\mathbb{L}}(\mathfrak{L})$ to $(\mathbf{G} \circ \mathbf{F})(\mathfrak{L})$.

Proof By definition $(\mathbf{G} \circ \mathbf{F})(\mathfrak{L}) = (\mathcal{L}_{\mathbf{F}(\mathfrak{L})}, \subseteq, \sim(\cdot))$. Define a function

$$\tau_{\mathfrak{L}}^{-1} : \mathcal{L}_{\mathbf{F}(\mathfrak{L})} \to L :: P \mapsto \bigvee P$$

For each $a \in L$ and $P \in \mathcal{L}_{\mathbf{F}(\mathfrak{L})}$, by Lemma 16 $P = [\![b]\!]$ for a $b \in L$, so by Lemma 2

$$(\tau_{\mathfrak{L}}^{-1} \circ \tau_{\mathfrak{L}})(a) = \tau_{\mathfrak{L}}^{-1}(\tau_{\mathfrak{L}}(a)) = \tau_{\mathfrak{L}}^{-1}([\![a]\!]) = \bigvee [\![a]\!] = a$$
$$(\tau_{\mathfrak{L}} \circ \tau_{\mathfrak{L}}^{-1})(P) = \tau_{\mathfrak{L}}(\tau_{\mathfrak{L}}^{-1}(P)) = \tau_{\mathfrak{L}}(\bigvee P) = [\![\bigvee P]\!] = [\![\bigvee [\![b]\!]]\!] = [\![b]\!] = P$$

Hence $\tau_{\mathfrak{L}}^{-1} \circ \tau_{\mathfrak{L}} = Id_L$ and $\tau_{\mathfrak{L}} \circ \tau_{\mathfrak{L}}^{-1} = Id_{\mathcal{L}_{\mathbf{F}(\mathfrak{L})}}$. Moreover, for any $a, b \in L$, by Lemma 2 we have the following equivalence:

$$\tau_{\mathfrak{L}}^{-1}([\![a]\!]) \leq \tau_{\mathfrak{L}}^{-1}([\![b]\!]) \Leftrightarrow \bigvee [\![a]\!] \leq \bigvee [\![b]\!] \Leftrightarrow a \leq b \Leftrightarrow [\![a]\!] \subseteq [\![b]\!] \Leftrightarrow \tau_{\mathfrak{L}}(a) \subseteq \tau_{\mathfrak{L}}(b)$$

We show that $\tau_{\mathfrak{L}}$ is an $\mathbb{L}$-morphism. By the above it is not hard to see that $\tau_{\mathfrak{L}}$ is a poset isomorphism from $(L, \leq)$ to $(\mathcal{L}_{\mathbf{F}(\mathfrak{L})}, \subseteq)$ such that, for any $a \in L$ and $A \subseteq L$,

- a is the infimum of A, if and only if $\tau_{\mathfrak{L}}(a)$ is the infimum of $\tau_{\mathfrak{L}}[A]$;
- $a \in At(\mathfrak{L})$, if and only if $\tau_{\mathfrak{L}}(a) \in At((\mathbf{G} \circ \mathbf{F})(\mathfrak{L}))$;
- $a \in coAt(\mathfrak{L})$, if and only if $\tau_{\mathfrak{L}}(a) \in coAt((\mathbf{G} \circ \mathbf{F})(\mathfrak{L}))$;

Hence, for any $A \subseteq L$, $\tau_{\mathfrak{L}}(\bigwedge A) = \bigcap \tau_{\mathfrak{L}}[A]$; for each $\{p\} \in At((\mathbf{G} \circ \mathbf{F})(\mathfrak{L}))$, $p \in At(\mathfrak{L})$ satisfies $\{p\} \subseteq \{p\} = [\![p]\!] = \tau_{\mathfrak{L}}(p)$; and, for each $a \in coAt(\mathfrak{L})$, $\tau_{\mathfrak{L}}(a) \in coAt((\mathbf{G} \circ \mathbf{F})(\mathfrak{L}))$ satisfies $\tau_{\mathfrak{L}}(a) \subseteq \tau_{\mathfrak{L}}(a)$. Therefore, $\tau_{\mathfrak{L}}$ is an $\mathbb{L}$-morphism.

Similarly, using the above equivalence we can show that $\tau_{\mathfrak{L}}^{-1}$ is an $\mathbb{L}$-morphism.

As a result, $\tau_{\mathfrak{L}}^{-1} \circ \tau_{\mathfrak{L}} = id_{\mathfrak{L}}$ and $\tau_{\mathfrak{L}} \circ \tau_{\mathfrak{L}}^{-1} = id_{(\mathbf{G} \circ \mathbf{F})(\mathfrak{L})}$. □

Lemma 37 τ satisfies naturality.

Proof Let $h : \mathfrak{L}_1 \to \mathfrak{L}_2$ be an $\mathbb{L}$-morphism. $(\mathbf{G} \circ \mathbf{F})(h) \circ \tau_{\mathfrak{L}_1} = \tau_{\mathfrak{L}_2} \circ h$, because

$$\begin{aligned}
\big((\mathbf{G} \circ \mathbf{F})(h) \circ \tau_{\mathfrak{L}_1}\big)(a_1) &= \mathbf{G}(\mathbf{F}(h))([\![a_1]\!]) \\
&= \mathrm{Ker}(\mathbf{F}(h)) \cup (\mathbf{F}(h))^{-1}[[\![a_1]\!]] \\
&= \{p_2 \in At(\mathfrak{L}_2) \mid \ell_h(p_2) \leq_1 a_1 \} \\
&= \{p_2 \in At(\mathfrak{L}_2) \mid p_2 \leq_2 h(a_1)\} \\
&= [\![h(a_1)]\!] \\
&= (\tau_{\mathfrak{L}_2} \circ h)(a_1)
\end{aligned}$$

holds for each $a_1 \in L_1$. □

Theorem 38 τ is a natural isomorphism from $1_{\mathbb{L}}$ to $\mathbf{G} \circ \mathbf{F}$.

Proof It follows from the previous two lemmas. □

Second we define η. For a quantum Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$, define a function

$$\eta_{\mathfrak{F}} : \Sigma \rightarrow At(\mathfrak{L}_{\mathfrak{F}}) :: s \mapsto \{s\}$$

Lemma 39 For each quantum Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$, $\eta_{\mathfrak{F}}$ is an isomorphism in $\mathbb{F}$ from $1_{\mathbb{F}}(\mathfrak{F})$ to $(\mathbf{F} \circ \mathbf{G})(\mathfrak{F})$.

Proof By definition $(\mathbf{F} \circ \mathbf{G})(\mathfrak{F}) = (At(\mathfrak{L}_{\mathfrak{F}}), \rightarrow_{\mathfrak{L}_{\mathfrak{F}}})$. Define a function

$$\eta_{\mathfrak{F}}^{-1} : At(\mathfrak{L}_{\mathfrak{F}}) \rightarrow \Sigma :: \{s\} \mapsto s$$

Note that $\eta_{\mathfrak{F}}^{-1} \circ \eta_{\mathfrak{F}} = Id_{\Sigma}$ and $\eta_{\mathfrak{F}} \circ \eta_{\mathfrak{F}}^{-1} = Id_{At(\mathfrak{L}_{\mathfrak{F}})}$. Moreover, for any $s, t \in \Sigma$,

$$\eta_{\mathfrak{F}}^{-1}(\{s\}) \rightarrow \eta_{\mathfrak{F}}^{-1}(\{t\}) \Leftrightarrow s \rightarrow t \Leftrightarrow \{s\} \not\subseteq \sim\{t\} \Leftrightarrow \{s\} \rightarrow_{\mathfrak{L}_{\mathfrak{F}}} \{t\} \Leftrightarrow \eta_{\mathfrak{F}}(s) \rightarrow_{\mathfrak{L}_{\mathfrak{F}}} \eta_{\mathfrak{F}}(t)$$

We show that $\eta_{\mathfrak{F}}$ is an $\mathbb{F}$-morphism. Assume that $\eta_{\mathfrak{F}}(s) = \{s\}$ and $\{s\} \rightarrow_{\mathfrak{L}_{\mathfrak{F}}} \{t\}$. Then, for any $w \in \Sigma$, by definition $\eta_{\mathfrak{F}}(w)$ is defined and by the above equivalence $t \rightarrow w \Leftrightarrow \{t\} \rightarrow_{\mathfrak{L}_{\mathfrak{F}}} \eta_{\mathfrak{F}}(w)$. Hence $(t, \{t\})$ satisfies $(\mathrm{Ad})_{\eta_{\mathfrak{F}}}$, so $\eta_{\mathfrak{F}}$ is an $\mathbb{F}$-morphism.

Similarly, using the above equivalence, we can show that $\eta_{\mathfrak{F}}^{-1}$ is an $\mathbb{F}$-morphism.

Therefore, $\eta_{\mathfrak{F}}^{-1} \circ \eta_{\mathfrak{F}} = id_{\mathfrak{F}}$ and $\eta_{\mathfrak{F}} \circ \eta_{\mathfrak{F}}^{-1} = id_{(\mathbf{F} \circ \mathbf{G})(\mathfrak{F})}$. □

Lemma 40 η satisfies naturality.

Proof Let $F : \mathfrak{F}_1 \rightarrow \mathfrak{F}_2$ be an $\mathbb{F}$-morphism, $s_1 \in \Sigma_1$ and $\{s_2\} \in At(\mathcal{L}_{\mathfrak{F}_2})$.

$$\begin{aligned}
& (s_1, \{s_2\}) \in (\mathbf{F} \circ \mathbf{G})(F) \circ \eta_{\mathfrak{F}_1} \\
\Leftrightarrow\ & (\{s_1\}, \{s_2\}) \in \mathbf{F}(\mathbf{G}(F)) \\
\Leftrightarrow\ & \{s_2\} = \ell_{\mathbf{G}(F)}(\{s_1\}) \\
\Leftrightarrow\ & \{s_2\} = \bigcap \{P_2 \in \mathcal{L}_{\mathfrak{F}_2} \mid \{s_1\} \subseteq \mathbf{G}(F)(P_2)\} \\
\Leftrightarrow\ & \{s_2\} = \bigcap \{P_2 \in \mathcal{L}_{\mathfrak{F}_2} \mid \{s_1\} \subseteq \mathrm{Ker}(F) \cup F^{-1}[P_2]\} \\
\Leftrightarrow\ & \{s_2\} = \bigcap \{P_2 \in \mathcal{L}_{\mathfrak{F}_2} \mid F(s_1) \text{ is undefined or } F(s_1) \in P_2\} \\
\Leftrightarrow\ & F(s_1) \text{ is defined and } \{s_2\} = \{F(s_1)\} = (\eta_{\mathfrak{F}_2} \circ F)(s_1) \\
\Leftrightarrow\ & (s_1, \{s_2\}) \in \eta_{\mathfrak{F}_2} \circ F
\end{aligned}$$

Therefore, $(\mathbf{F} \circ \mathbf{G})(F) \circ \eta_{\mathfrak{F}_1} = \eta_{\mathfrak{F}_2} \circ F$. □

Theorem 41 η is a natural isomorphism from $1_{\mathbb{F}}$ to $\mathbf{F} \circ \mathbf{G}$.

Proof It follows from the previous two lemmas. □

From the two theorems in this section we conclude the following:

Theorem 42 $(\mathbf{F}, \mathbf{G}, \tau, \eta)$ forms a duality between $\mathbb{L}$ and $\mathbb{F}$.

5 Conclusion and Future Work

In this paper, we define the category $\mathbb{L}$ of Piron lattices and the category $\mathbb{F}$ of quantum Kripke frames, and show that they are dual to each other. We emphasize that the objects and the arrows in both categories are not merely abstract mathematical notions but of significance in physics. Considering that the propositional logic of Piron lattices and that of Hilbert lattices have been extensively studied in the literature, future work will be more about the category of quantum Kripke frames. There are many interesting research topics, and here I briefly mention two.

One is the modal axiomatization of quantum Kripke frames in Kripke semantics. This topic is interesting because of two observations about the literature, e.g. [6] and [7]. One is that the relational semantics of quantum logic in these papers only involves Kripke frames which are much more general than quantum Kripke frames. The other is that the relational semantics in these papers is only for basic propositional logic. Now that our duality result and Piron's theorem demonstrate the significance of quantum Kripke frames in physics, it will be interesting to axiomatize them in a propositional modal logic using the sophisticated theory of Kripke semantics. Moreover, the Kripke modalities of the non-orthogonality relation are worth studying because, combined with (classical) negation, they can define many interesting logical notions, for example, the quantum negation, i.e. orthocomplement, and the Sasaki hook, which is the counterpart in quantum logic of material implication and is a Stalnaker conditional in the formal semantics [8]. Finally, since existence of approximation is a second-order condition, modal axiomatization of quantum Kripke frames is technically challenging and will stimulate the development of modal logic.

The other is the logical study of continuous homomorphisms. Note that the definition of these partial functions does not rely on the properties of quantum Kripke frames but makes sense in Kripke frames in general. Moreover, continuous homomorphisms are different from bisimulations, which essentially are the truth-preserving relations for the Kripke modalities (Theorem 2.62 in [5]). On the one hand, a bisimulation may not be a partial function, and thus may not be a continuous homomorphism; on the other hand, a continuous homomorphism F may not be a bisimulation, because one can prove that, when both Kripke frames are symmetric, F is a bisimulation, if and only if $(s_1, s_2) \in F \Leftrightarrow (s_1, s_2)$ satisfies (Ad)$_F$. Therefore, it will be interesting to devise a modality on Kripke frames such that continuous homomorphisms are exactly the truth-preserving relations, and to investigate the logics of this modality for quantum Kripke frames and for Kripke frames in general.[9] Since these partial functions are generalizations of bounded linear maps between Hilbert spaces and thus are significant in physics, such a modality with motivation from physics will be an

[9]I am very grateful to Dr. Nick Bezhanishvili and Dr. Yanjing Wang, for in two different occasions they raised this topic to me.

interesting topic and enrich the study of modal logic.

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量子逻辑中一个形式化的状态–性质对偶关系

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摘　要

本文展示了量子物理中一个状态—性质对偶关系的形式化。在性质方面，Piron 证明了 Piron 格（最初被称为不可分解的命题系统）刻画了量子系统的可测试性质所组成的结构。在状态方面，我们定义量子 Kripke 框架来刻画量子系统的状态在非正交关系之下所组成的结构。而且，我们定义了 Piron 格之间的线性态射，并把 Piron 格所组成的类组织成一个范畴。我们也定义了量子 Kripke 框架之间的连续同态，并把量子 Kripke 框架所组成的类组织成一个范畴。最后，我们证明了在范畴论的意义上 Piron 格所组成的范畴和量子 Kripke 框架所组成的范畴是对偶的，这样我们就用数学的语言描述了量子物理里面一个直观上的状态—性质对偶关系。这个形式化的对偶关系在代数结构和关系结构之间建立了联系，这将会有助于研究关于量子物理的逻辑。