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A logic of goal-directed knowing how

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Abstract In this paper, we propose a decidable single-agent modal logic for reasoning about goal-directed "knowing how", based on ideas from linguistics, philosophy, modal logic, and automated planning in AI. We first define a modal language to express "I know how to guarantee φ given ψ" with a semantics based not on standard epistemic models but on labeled transition systems that represent the agent's knowledge of his own abilities. The semantics is inspired by conformant planning in AI. A sound and complete proof system is given to capture valid reasoning patterns, which highlights the compositional nature of "knowing how". The logical language is further extended to handle knowing how to achieve a goal while maintaining other conditions.

Keywords Knowing how · Epistemic logic · Conformant planning · Modal logic

1 Introduction

Von Wright (1951) and Hintikka (1962) laid out the syntactic and semantic foundations of epistemic logic respectively in their seminal works. The standard picture of epistemic logic usually consists of: a propositional modal language which can express "an agent knows that φ"; a Kripke semantics that incarnates the slogan "*knowledge (information) as elimination of uncertainty*"; a proof system that syntactically characterizes a normal modal logic somewhere between $\mathbb{S}4$ and $\mathbb{S}5$ subjective to different opinions about the so-called "introspection axioms". Despite the suspicions from philosophers

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in its early days [cf. (Lenzen 1978)], the past half-century has witnessed the blossom of this logical investigation of propositional knowledge with applications in epistemology, theoretical computer science, artificial intelligence, economics, and many other disciplines besides its birth place of modal logic [see, e.g., (Wang 2011, Chap. 2) and (van Ditmarsch et al. 2015)].

However, the large body of research on epistemic logic mainly focuses on propositional knowledge expressed by "knowing that φ", despite the fact that in everyday life knowledge is expressed by also "knowing how", "knowing why", "knowing what", "knowing whether", and so on (knowing-wh below for brevity). Linguistically, these expressions of knowledge share the common form consisting of the verb "know" followed by some embedded wh-questions.[1] It is natural to assign a high-level uniform truth condition for these knowledge expressions in terms of knowing an answer to the corresponding question [cf. e.g., (Harrah 2002)]. In fact, already in the early days of epistemic logic, Hintikka (1962) had elaborate discussions on knowing-wh and its relation with questions in terms of first-order modal logic, which also shaped his later work on *Socratic Epistemology* (Hintikka 2007). For example, "knowing who Frank is" is rendered as $\exists x \mathcal{K}(Frank = x)$ by Hintikka (1962). See Wang (forthcoming) for a survey on Hintikka's contributions on this topic. However, partly because of the then-infamous philosophical and technical issues regarding the foundation of first-order modal logic (largely due to Quine), the development of epistemic logics beyond "knowing that" was hindered.[2] In the seminal work by Fagin et al. (1995), the first-order epistemic logic is just briefly touched without specific discussion of those expressions using different embedded questions. A promising recent approach is based on *inquisitive semantics* where propositions may have both informative content and inquisitive content [cf. e.g., (Ciardelli et al. 2013)]. An inquisitive epistemic logic which can handle "knowing that" and "knowing whether" is proposed by Ciardelli and Roelofsen (2015).

Departing from the linguistically motivated compositional analysis on knowing-wh, some researchers took a knowing-wh construction as a whole, and introduce a new modality instead of breaking it down by allowing quantifiers, equality and other logical constants to occur freely in the language [cf. (Plaza 1989; Hart et al. 1996; van der Hoek and Lomuscio 2003)]. For example, "knowing what a password is" is rendered by "$Kv\ password$" by Plaza (1989) instead of $\exists x \mathcal{K}\ password = x$, where Kv is the new modality. This move seems promising since by restricting the language we may avoid some philosophical issues of first-order modal logic, retain the decidability of the logic, and focus on special logical properties of each particular knowing-wh construction at a high abstraction level. A recent line of work results from this idea

[1] There is a cross-lingual fact: such knowing-wh sentences become ungrammatical if the verb "know" is replaced by "believe", e.g., I believe how to swim. This may shed some shadow on philosophers' usual conception of knowledge in terms of strengthened belief. Linguistically, this phenomenon occurs to many other verbs which can be roughly categorized using factivity, cf., e.g, (Egrè 2008).

[2] Nevertheless Hintikka addressed some of those issues about first-order modal logic insightfully in the context of epistemic logic, see, e.g., a wonderful survey paper by Hintikka (1989). Many of those issues are also elegantly addressed in intensional first-order modal logic, cf. e.g., (Fitting and Mendelsohn 1998). There is a wonderful survey on quantified epistemic logic by Gochet and Gribomont (2006).

(Wang and Fan 2013, 2014; Fan et al. 2014, 2015; Gu and Wang 2016).[3] The resulting logics are usually not normal.[4] Moreover, a common technical difficulty in such an approach is the apparent asymmetry of syntax and semantics: the modal language is relatively weak compared to the models which contain enough information to facilitate an intuitive semantics of knowing-wh, and this requires new techniques.

1.1 Knowing how

Among all the knowing-wh expressions, the most discussed one in philosophy and AI is "knowing how". Indeed, it sounds the most distant from propositional knowledge (knowledge-that): knowing how to swim seems distinctly different from knowing that it is raining outside. One question that keeps philosophers busy is whether knowledge-how (the knowledge expressed by "knowing how") is reducible to knowledge-that. Here philosophers split into two stances: the intellectualists who think knowledge-how is a subspecies of knowledge-that [e.g., (Stanley and Williamson 2001)], and the anti-intellectuallists who do not think so [e.g., (Ryle 1946)]. The anti-intellectualism may win your heart at the first glance by equating knowledge-how to certain ability, but the linguistically and logically well-versed intellectualists may have their comebacks at times (think about the previously mentioned interpretation of knowing-wh as knowing an answer).[5] In AI, starting from the early days (McCarthy and Hayes 1969; McCarthy 1979; Moore 1985), people have been studying about representation and reasoning of *knowledge and action* which is often treated as synonym for knowledge-how in AI, in particular about procedural knowledge based on specifiable plans or strategies such as coming out of a maze or winning a game. However, there is no common consensus on how to capture the logic of "knowing how" formally [cf. the excellent surveys by Gochet (2013) and Ågotnes et al. (2015)].

One problem that bothers logicians is that simply combining "knowing that" and "ability" does not lead to a genuine notion of "knowing how" as discussed by Herzig (2015). For example, adding knowledge operator to alternating-time temporal logic by Alur et al. (2002) can express knowing that there is a strategy for some goal. However it is a *de dicto* reading of know-how in the shape of $\mathcal{K}\exists x\varphi(x)$, not the desired *de re* reading: $\exists x\mathcal{K}\varphi(x)$. As Herzig (2015) remarked, people proposed different solutions: we may specify the x in the modality [e.g., knowing that executing *abc* will make sure φ (Herzig et al. 2013; Belardinelli 2014)]; we can also try to insert $\mathcal{K}$ in-between the existential quantifier and the ability modality where epistemic see-to-that-it (STIT) logic may help [cf. e.g., (Broersen and Herzig 2015)].

In this paper, along the line of our previous works, we take "knowing how" as a *single* modality and present a new attempt to formalize an important kind of "knowing

[3] See (Wang forthcoming) for a survey with discussions of related work on quantified epistemic logic.

[4] For example, *knowing whether* $p \to q$ and knowing whether p together does not entail knowing whether q. Likewise, *knowing how* to p and knowing how to q does not entail knowing how to $p \wedge q$. Moreover, you may not *know why* a tautology is a tautology which contradicts necessitation.

[5] Fantl (2008) presents a survey of the debate. A comprehensive collection of the related papers (200^+) can be found at http://philpapers.org/browse/knowledge-how, edited by John Bengson.

how" and lay out its logic foundation, inspired by the aforementioned perspectives of linguistics, philosophy, and AI.

Some clarifications have to be made first:

– We will focus on the logic of *goal-directed* "knowing how" as Gochet (2013) puts it, such as knowing how to prove a theorem, how to open the door, how to bake a cake, and how to cure the disease, i.e., linguistically, mainly about knowing how followed by an *achievement verb* or an *accomplishment verb* according to the classification of Vendler (1967).[6] On the other hand, we will not talk about the following "knowing how": I know how the computer works (explanation), I know how happy she is (degree), I know how to speak English (rule-directed) and so on. In the later part, we will also address goal-directed know-how while maintaining some conditions, e.g., knowing how to be rich without breaking the law.
– The goal of this paper is *not* to address the philosophical debates between intellectualism and anti-intellectualism.[7] However, to some extent, we are inspired by the ideas from both philosophical stances.
– We focus on the single-agent case without probability and any measure on the efforts and expertise in actions.

1.2 Basic ideas behind the syntax and semantics

Different from the cases on "knowing whether" and "knowing what", there is nothing close to a consensus on what would be the syntax and semantics of *the* logic of "knowing how". Various attempts were made using Situation Calculus, ATL, or STIT logic to express different versions of "knowing how", cf. e.g., (Moore 1985; Morgenstern 1986; Herzig and Troquard 2006; Broersen 2008; Jamroga and van der Hoek 2004; Gochet 2013). However, as we mentioned before, we do not favor a syntactically compositional analysis using powerful quantified logical languages. Instead, we would like to take the "knowing how" construction as a single (and only) modality in our language. It seems natural to introduce a modality $\mathcal{K}h\varphi$ to express the goal-directed "knowing how to achieve the goal φ". It sounds similar to "having the ability to achieve the goal φ", as many anti-intellectualists would agree. It seems harmless to go one step further as in the AI literature to interpret this type of "knowing how" as that the agent *can* achieve φ. However, it is crucial to note the following problems of such an anti-intelectualistic ability account:

1. Knowing how to achieve a goal may not entail that you *can* realize the goal now. For example, as intellectualists would remark, a broken-arm pianist may still know how to play piano even if he cannot play right now, and a chef may still know how

[6] Here knowing how to do an activity (like swimming) is *not* a typical example for our treatment, although we hope our formalism captures some common features shared also by them. As discussed in Gochet (2013), "knowing how" plus activities, though more philosophically interesting, is less demanding in logical structure than others.

[7] The philosophical implication of a similar treatment is discussed by Lau and Wang (2016), where an intellectualistic account is advocated, which may help to reconcile intellectualism and anti-intellectualistic ability account.

to make cakes even when the sugar is run out [cf. e.g., (Stanley and Williamson 2001)].
2. Even when you have the ability to win a lottery by luckily buying the right ticket (and indeed win it in the end), it does not mean you know how to win the lottery, since you cannot *guarantee* the result [cf. e.g., Carr (1979)].

To reconcile our intuition about the ability involved in "knowing how" and the first problem above, it is observed by Lau and Wang (2016) that "knowing how" expressions in context often come with implicit preconditions.[8] For example, if you ask someone local in Amsterdam: "Do you know how to go to the airport?" he would probably say "Yes". However, if you then tell him that all the public transportation methods are down due to some strike, then maybe his would say "Then I don't know". This shows that the first claim of "knowing how" assumed some implicit precondition: e.g., the public transportation is still running. Likewise, it sounds all right to say that you know how to bake a cake even when you do not have all the ingredients right now: you can do it *given that* you have all the ingredients. In our logical language, we make such context-dependent preconditions explicit by introducing the binary modality $\mathcal{K}h(\psi, \varphi)$ expressing that the agent knows how to achieve φ given the precondition ψ.[9] Actually, we introduced a similar conditional "knowing what" operator in Wang and Fan (2014) to capture the conditional knowledge such as "I would know my password for this website, given it is 4-digit" (since I only have one 4-digit password ever).[10] Wang and Fan (2014) showed that this conditionaliztion proved to be also useful to encode the potential dynamics of knowledge. In the later part of the paper, we generalize the operator into a ternary one: $\mathcal{K}h(\psi, \chi, \varphi)$, read as I know how to achieve φ given ψ while maintaining χ. We will come back to the details later.

Now, to reconcile the intuition of ability-based know-how with the second problem above, we need to interpret the ability more precisely to exclude the lucky draws. Our main idea comes from *conformant planning* in AI which is exactly about *how* to achieve a goal by a plan (sequence of actions) which can never fail given some initial uncertainty [cf. e.g., Smith and Weld (1998)]. The word "conformant" means that the plan should always work given the uncertainty and no changes are allowed during the execution. For example,[11] consider the following map of a floor in a building, and suppose that you only know that you are at a p-place but do not know exactly where you are. Do you know how to reach a safe place (marked by q)?

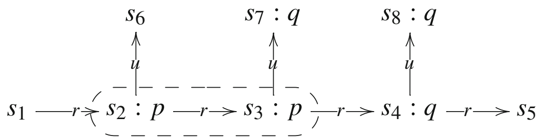
8 Such conditions are rarely discussed in the philosophical literature of "knowing how" with few exceptions such as Noë (2005).

9 By using the condition, one can be said to know better how to swim than another if he can do it in a more hostile environment (thus weakening the condition) see Lau and Wang (2016).

10 Such conditionals are clearly not simple (material) implications and they are closely related to conditional probability and conditional belief [cf. e.g., Tillio et al. (2014)].

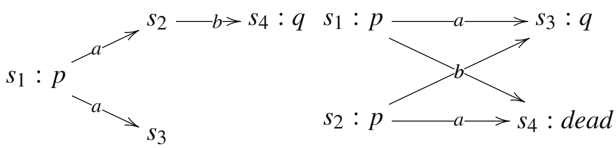
11 Taken from Wang and Li (2012), Yu et al. (2016).

Example 1



It is not hard to see that there exists a plan to *guarantee* your safety from *any* place marked by p, which is to move r first then move u. On the contrary, the plan rr and the plan u might fail depending on where you actually are. The locations in the map can be viewed as states of affairs and the labelled directional edges between the states can encode your own "knowledge" of the available actions and their effects.[12] Intuitively, to know how to achieve φ requires that you can guarantee φ. Consider the following examples which represent the agent's knowledge about his own abilities.

Example 2



The graph on the left denotes that you know you can do a at the p-state s_1 but you are not sure what the consequence is: it may lead to either s_2 or s_3, and the exact outcome is out of your control. Therefore, this action is *non-deterministic* to you. In this case, ab is not a good plan since it may fail to be executable. Thus it sounds unreasonable to claim that you know how to reach q given p.

Now consider the graph on the right. Let a and b be two medicines to cure the same symptom p depending on the exact cause (s_1 or s_2). Unfortunately, using in wrong circumstances the medicines will cause the death of the patient. As a doctor, it is indeed true that you can cure the patient (to achieve q) if you are told the exact cause. However, responsible as a doctor, can you say you know how to cure the patient given only the information of the symptom p? Definitely not. These planning examples suggest the following truth condition for the modal formula $\mathcal{K}h(\psi, \varphi)$ w.r.t. a graph-like model representing the agent's knowledge about his or her abilities in terms of available actions and their possibly non-deterministic effects: There *exists* a sequence σ of actions such that from *all* the ψ-states in the graph, σ will *always* succeed in reaching φ-states.

[12] The agent may have more abilities *de facto* than what he may realize. It is important to make sure the agent can *knowingly* guarantee the goal in terms of the ability he is aware of, cf. (McCarthy and Hayes 1969; Broersen 2008; Ågotnes et al. 2015).

Note that the nesting structure of quantifiers in the above truth condition is $\exists\forall\forall$.[13] The first $\exists$ fixes a unique plan, the first $\forall$ checks all the possible states satisfying the condition ψ, and the second $\forall$ make sure the goal is guaranteed.

There are several points to be highlighted:

- $\exists$ cannot be swapped with the first $\forall$: see the discussion about the second graph in Example 2, which amounts to the distinction between *de re* and *de dicto* in the setting of "knowing how" [cf. also (Moore 1985; Jamroga and van der Hoek 2004; Herzig and Troquard 2006; Ågotnes et al. 2015) and uniform strategies in imperfect information games].
- There is *no* explicit "knowing that" in the above truth condition, which differs from the formalization $\exists x\mathcal{K}\varphi(x)$ advocated by intellectualism [e.g., (Stanley 2011)].[14] However, you may view the first $\forall$ as a knowledge operator which is essentially a universal quantifier over epistemic alternatives (Hintikka 1962). Indeed, the condition ψ gives a range of epistemic possibilities.
- The truth condition is based on a Kripke-like model without epistemic relations, as in the treatment of (imperfect) procedure information by Wang (2015b). The graph model represents the agent's knowledge (or database) of his actions and their effects. As it will become more clear later on, although the knowing how logic is apparently not normal, it is not necessary to go for neighbourhood or topological models to accommodate such a non-normal modal logic, given that the truth condition of the modality is non-standard [cf. also Kracht and Wolter (1997)];
- Finally, our interpretation of "knowing how" does not fit the standard epistemic scheme "knowledge as elimination of uncertainty" proposed by Hintikka (1962). It is not about possible worlds indistinguishable from the "real world". The truth of $\mathcal{K}h(\psi,\varphi)$ does not depend on whether the current world satisfying ψ: the semantics will let us jump to all the ψ-worlds. As we mentioned, this is to handle the cases when people have knowledge-how but cannot realize the goal currently. In sum, "knowing how" is *global* in nature and it is not a typical contingent knowledge of the current world (e.g., I know that Johan happen to be sitting in his chair).

In Sect. 2, we flesh out the above ideas in precise definitions: first a simple formal language, then the semantics based on the idea of conformant planning, and finally a proof system which is sound and complete, as proved in Sect. 3. The canonical model construction in Sect. 3 also gives us the small model property (and thus the decidability) of our logic. In Sect. 4, we generalize our modality to a ternary one to handle the properties that we want to maintain while reaching a goal. We also introduce the announcement operator into the language to capture updates of background knowledge. However, the new operator is not reducible in this language. In the last section, we summarize our new ideas beyond the standard schema of epistemic logic, and point out many future directions.

[13] Brown (1988) introduced a modality for *can* φ with the following $\exists\forall$ schema over neighbourhood models: *there is* a relevant cluster of possible worlds (as the outcomes of an action) where φ is true in *all* of them.

[14] This also distinguishes this work from our earlier philosophical discussion in Lau and Wang (2016), where intellectualism was defended by giving an $\exists x\mathcal{K}\varphi(x)$-like truth condition informally.

2 The framework

As discussed before, we make the implicit precondition in "knowing how" explicit in the logical language.

Definition 1 Given a set of proposition letters $\mathbf{P}$, the language $\mathbf{L_{Kh}}$ is defined as follows:

$$\varphi \quad ::= \quad \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathcal{K}h(\varphi, \varphi)$$

where $p \in \mathbf{P}$. $\mathcal{K}h(\psi, \varphi)$ expresses that the agent knows how to achieve φ given ψ. We use the standard abbreviations $\bot$, $\varphi \vee \psi$ and $\varphi \to \psi$, and define $\mathcal{U}\varphi$ as the abbreviation of $\mathcal{K}h(\neg\varphi, \bot)$. $\mathcal{U}$ is intended to be a *universal modality* as it will become more clear after defining the semantics.

As we mentioned, in our models, there is no epistemic relation, but only the action labeled transitions which are used to capture the agent's abilities.

Definition 2 Given the set of proposition letters $\mathbf{P}$ and a countable non-empty set of action symbols Σ. An *ability map* is essentially a labeled transition system $(\mathcal{S}, \mathcal{R}, \mathcal{V})$ where:

- $\mathcal{S}$ is a non-empty set of states;
- $\mathcal{R} : \Sigma \to 2^{\mathcal{S}\times\mathcal{S}}$ is a collection of transitions labeled by actions in Σ;
- $\mathcal{V} : S \to 2^{\mathbf{P}}$ is a valuation function.

We write $s \xrightarrow{a} t$ if $(s, t) \in \mathcal{R}(a)$. For a sequence $\sigma = a_1 \dots a_n \in \Sigma^*$, we write $s \xrightarrow{\sigma} t$ if there exist $s_2 \dots s_n$ such that $s \xrightarrow{a_1} s_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} s_n \xrightarrow{a_n} t$. Note that σ can be the empty sequence ϵ (when $n = 0$), and we set $s \xrightarrow{\epsilon} s$ for any s. Let σ_k be the initial segment of σ up to a_k for $k \leq |\sigma|$. In particular let $\sigma_0 = \epsilon$. We say that σ is *strongly executable* at s if: $\sigma = \epsilon$, or $\sigma = a_1 \dots a_n$ and for any $0 \leq k < n$ and any t, $s \xrightarrow{\sigma_k} t$ implies that t has at least one a_{k+1}-successor. It is not hard to see that if σ is strongly executable at s then it is executable at s, i.e., $s \xrightarrow{\sigma} t$ for some t.

Note that, according to our above definition, ab is not strongly executable from s_1 in the left-hand-side model of Example 2, since s_3 has no b-successor but it can be reached from s_1 by $a = (ab)_1$.

We want to stress that the symbols in Σ do not appear in the formal language $\mathbf{L_{Kh}}$ and they can vary in different models, in contrast with the usual language of dynamic logic. However, the following semantics will make use of Σ, which demonstrates the asymmetry between syntax and semantics as we discussed.

Definition 3 (Semantics of $\mathbf{L_{Kh}}$)

$$\begin{array}{rcl}
\mathcal{M}, s \vDash \top & & always \\
\mathcal{M}, s \vDash p & \Leftrightarrow & p \in V(s) \\
\mathcal{M}, s \vDash \neg\varphi & \Leftrightarrow & \mathcal{M}, s \nvDash \varphi \\
\mathcal{M}, s \vDash \varphi \wedge \psi & \Leftrightarrow & \mathcal{M}, s \vDash \varphi \text{ and } \mathcal{M}, s \vDash \psi \\
\mathcal{M}, s \vDash \mathcal{K}h(\psi, \varphi) & \Leftrightarrow & \text{there exists a } \sigma \in \Sigma^* \text{ such that for all } s' \text{ such that } \mathcal{M}, s' \vDash \psi : \\
 & & \sigma \text{ is strongly executable at } s' \text{ and for all } t \text{ such that } s' \xrightarrow{\sigma} t, \mathcal{M}, t \vDash \varphi
\end{array}$$

Note that the modality $\mathcal{K}h$ is *not local* in the sense that its truth does not depend on the designated state where it is evaluated. Thus it either holds on all the states or none of them. It is not hard to see that the schema of $\exists\forall\forall$ appears in the truth condition for $\mathcal{K}h$ where the last $\forall$ actually is a conjunction of two universally quantified parts: The strong executability (there is a $\forall$ in its definition) and the guarantee of the goal. These two together make sure the plan will never fail to achieve φ. It is a simple exercise to see that $\mathcal{K}h(p, q)$ holds in the model of Example 1, but not in the models of Example 2.

Now we can see that the operator $\mathcal{U}$ defined by $\mathcal{K}h$ is indeed a *universal modality*:[15]

$$\boxed{\mathcal{M}, s \vDash \mathcal{U}\varphi \Leftrightarrow \text{ for all } t \in \mathcal{S}, \mathcal{M}, t \vDash \varphi}$$

To see this, check the following:

$$\boxed{\begin{array}{lll}
\mathcal{M}, s \vDash \mathcal{K}h(\neg\psi, \bot) & \Leftrightarrow & \text{there exists a } \sigma \in \Sigma^* \text{ such that for every } \mathcal{M}, s' \vDash \neg\psi : \\
& & \sigma \text{ is strongly executable at } s' \text{ and if } s' \xrightarrow{\sigma} t \text{ then } \mathcal{M}, t \vDash \bot \\
& \Leftrightarrow & \text{there exists a } \sigma \in \Sigma^* \text{ such that for every } \mathcal{M}, s' \vDash \neg\psi : \\
& & \sigma \text{ is strongly executable at } s' \text{ and there is no } t \text{ such that } s' \xrightarrow{\sigma} t \\
& \Leftrightarrow & \text{there exists a } \sigma \in \Sigma^* \text{ such that for every } \mathcal{M}, s' \vDash \neg\psi : \bot \text{ holds} \\
& \Leftrightarrow & \text{there exists a } \sigma \in \Sigma^* \text{ such that there is no } s' \text{ such that } \mathcal{M}, s' \nvDash \psi \\
& \Leftrightarrow & \text{for all } t \in \mathcal{S}, \mathcal{M}, t \vDash \psi
\end{array}}$$

Here are some useful valid formulas according to the semantics.

Proposition 1 *The following are valid:*

$$\begin{array}{llll}
1 & \mathcal{U}p \wedge \mathcal{U}(p \to q) \to \mathcal{U}q & 2 & \mathcal{K}h(p, r) \wedge \mathcal{K}h(r, q) \to \mathcal{K}h(p, q) \\
3 & \mathcal{U}(p \to q) \to \mathcal{K}h(p, q) & 4 & \mathcal{U}p \to p \\
5 & \mathcal{K}h(p, q) \to \mathcal{U}\mathcal{K}h(p, q) & 6 & \neg\mathcal{K}h(p, q) \to \mathcal{U}\neg\mathcal{K}h(p, q)
\end{array}$$

Proof Since $\mathcal{U}$ is simply a universal modality, 1 and 4 are obvious. 3 is due to the fact that ϵ is allowed as a trivial plan. 5 and 6 are due to the fact that $\mathcal{K}h$ is global. The only non-trivial case is 2. Note that if there is a strongly executable sequence σ leading you from any p-state to some r-state, and there is a strongly executable sequence η from r-states to q-states, then $\sigma\eta$ is strongly executable from any p-state and it will make sure that you end up with q-states from any p-state. □

The validity of (2) above actually captures the intuitive sequential compositionality of "knowing how", as desired. Note that $\mathcal{K}h(p, q) \wedge \mathcal{K}h(p, r) \to \mathcal{K}h(p, q \wedge r)$ is not valid, e.g., knowing how to open the door and knowing how to close the door does not mean knowing how to open and close the door at the same time.

Based on the above axioms, we propose the following proof system $\mathbb{SKH}$ for $\mathbf{L_{Kh}}$ (where $\varphi[\psi/p]$ is obtained by uniformly substituting p in φ by ψ):

[15] Note that $\mathcal{U}$ is a very powerful modality in its expressiveness when combined with the standard $\Box$ modality, cf. (Goranko and Passy 1992).

System $\mathbb{SKH}$

Axioms		**Rules**	
TAUT	all axioms of propositional logic	MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
DISTU	$\mathcal{U}p \land \mathcal{U}(p \to q) \to \mathcal{U}q$	NECU	$\dfrac{\varphi}{\mathcal{U}\varphi}$
COMPKh	$\mathcal{K}h(p, r) \land \mathcal{K}h(r, q) \to \mathcal{K}h(p, q)$	SUB	$\dfrac{\varphi(p)}{\varphi[\psi/p]}$
EMPKh	$\mathcal{U}(p \to q) \to \mathcal{K}h(p, q)$		
TU	$\mathcal{U}p \to p$		
4KU	$\mathcal{K}h(p, q) \to \mathcal{U}\mathcal{K}h(p, q)$		
5KU	$\neg\mathcal{K}h(p, q) \to \mathcal{U}\neg\mathcal{K}h(p, q)$		

Proposition 1 plus some reflection on the usual inference rules should establish the soundness of $\mathbb{SKH}$. For completeness, we first get a taste of the deductive power of $\mathbb{SKH}$ by proving the following formulas which play important roles in the later completeness proof. In the rest of the paper we use $\vdash$ to denote $\vdash_{\mathbb{SKH}}$.

Proposition 2 *We can derive the following in* $\mathbb{SKH}$ *(names are given to be used later):*

TRI	$\mathcal{K}h(p, p)$
UKh	$\mathcal{U}(p \to r) \land \mathcal{U}(o \to q) \land \mathcal{K}h(r, o) \to \mathcal{K}h(p, q)$
4U	$\mathcal{U}p \to \mathcal{U}\mathcal{U}p$
5U	$\neg\mathcal{U}p \to \mathcal{U}\neg\mathcal{U}p$
COND	$\mathcal{K}h(\bot, p)$
UCONJ	$\mathcal{U}(\varphi \land \psi) \leftrightarrow (\mathcal{U}\varphi \land \mathcal{U}\psi)$.
PREKh	$\mathcal{K}h(\mathcal{K}h(p, q) \land p, q)$.
POSTKh	$\mathcal{K}h(r, \mathcal{K}h(p, q) \land p) \to \mathcal{K}h(r, q)$

Moreover, the following rule NECKh *is admissible:* $\vdash \varphi \implies \vdash \mathcal{K}h(\psi, \varphi)$.

Proof TRI is proved by applying NECU to the tautology $p \to p$, followed by EMPKh. UKh says if you weaken the goal and strengthen the condition you still know how. It is proved by applying EMPKh to $\mathcal{U}(p \to r)$ and $\mathcal{U}(o \to q)$ followed by applying COMPKh twice. 4U and 5U are special cases of 4KU and 5KU respectively since $\mathcal{U}\psi := \mathcal{K}h(\neg\psi, \bot)$. Since $\bot \to p$ is a tautology, we can apply NECU and EMPKh to obtain COND. UCONJ is a standard exercise for normal modality $\mathcal{U}$. Interestingly, PREKh says that you know how to guarantee q given both p and the fact that you know how to guarantee q given p. It can be proved by distinguishing two cases: $\mathcal{K}h(p, q)$ and $\neg\mathcal{K}h(p, q)$, and use COND and UKh respectively under the help of NECU. POSTKh can be proved easily based on COMPKh and PREKh. It says that you know how to achieve q given r if you know how to achieve a state where you know how to continue to achieve q.[16] Finally NECKh is the necessitation rule for $\mathcal{K}h$ which can be derived

[16] This is an analog of a requirement of "knowing how" by Moore (1985, p. 58): you need to make sure by doing the first step you will know how to continue.

by starting with the tautology $\psi \to \varphi$ (given $\vdash \varphi$) followed by the applications of NECU and EMPKh. □

Remark 1 From the above proposition and the system $\mathbb{SKH}$, we see that $\mathcal{U}$ is indeed an $\mathbb{S}5$ modality which can be considered as a version of "knowing that": you know that φ iff it holds on all the relevant possible states under the current restriction of attention (not just the epistemic alternatives to the actual one). The difference is that here the knowledge-that expressed by $\mathcal{U}\varphi$ refers to the "background knowledge" that you take for granted for now such as the knowledge of the laws of physics, rather than the contingent knowledge such as "he is sitting on the chair", which can be made false when some action happens. Another interesting thing to notice is that UKh actually captures an important connection between "knowing that" and "knowing how", e.g., you know how to cure a disease if you know that it is of a certain type and you know how to cure this type of the disease in general. We will come back to the relation between "knowing how" and "knowing that" at the end of the paper.

It is crucial to establish the following replacement rule to ease the later proofs.

Proposition 3 *The replacement of equivalents ($\vdash \varphi \leftrightarrow \psi \implies \vdash \chi \leftrightarrow \chi[\psi/\varphi]$) is an admissible rule in $\mathbb{SKH}$, where the substitution $[\psi/\varphi]$ may apply to some (not necessarily all) of the occurrences.*

Proof It becomes a standard exercise in modal logic if we can prove that the following two rules are admissible in $\mathbb{SKH}$:

$$\vdash \psi \leftrightarrow \varphi \implies \vdash \mathcal{K}h(\psi, \chi) \leftrightarrow \mathcal{K}h(\varphi, \chi), \quad \vdash \psi \leftrightarrow \varphi \implies \vdash \mathcal{K}h(\chi, \psi) \leftrightarrow \mathcal{K}h(\chi, \varphi).$$

Actually we can derive (all the instances of) these two rules as follows (we only show the first one since the second one is similar.):

1	$\psi \leftrightarrow \varphi$	assumed
2	$\varphi \to \psi$	1, TAUT
3	$\mathcal{U}(\varphi \to \psi) \wedge \mathcal{U}(\chi \to \chi)$	NECU
4	$\mathcal{U}(\varphi \to \psi) \wedge \mathcal{U}(\chi \to \chi) \wedge \mathcal{K}h(\psi, \chi) \to \mathcal{K}h(\varphi, \chi)$	UKh, SUB
5	$\mathcal{K}h(\psi, \chi) \to \mathcal{K}h(\varphi, \chi)$	MP(3, 4)
6	$\mathcal{K}h(\varphi, \chi) \to \mathcal{K}h(\psi, \chi)$	symmetric version of 2–5
7	$\mathcal{K}h(\psi, \chi) \leftrightarrow \mathcal{K}h(\varphi, \chi)$	TAUT

□

In the rest of the paper we often use the above rule of replacement implicitly.

3 Completeness and decidability

Proposition 1 established the soundness of $\mathbb{SKH}$ and in this section we show the completeness.

Given a set of $\mathbf{L_{Kh}}$ formulas Δ, let $\Delta|_{\mathcal{K}h}$ be the collection of the $\mathcal{K}h$-formulas in it:

$$\Delta|_{\mathcal{K}h} = \{\mathcal{K}h(\psi, \varphi) \mid \mathcal{K}h(\psi, \varphi) \in \Delta\}.$$

Similarly, let $\Delta|_{\neg\mathcal{K}h}$ be the following collection:

$$\Delta|_{\neg\mathcal{K}h} = \{\neg\mathcal{K}h(\psi, \varphi) \mid \neg\mathcal{K}h(\psi, \varphi) \in \Delta\}.$$

Note that the semantics of $\mathcal{K}h$ formulas does not depend on the current state, thus if they are true then they are true everywhere in the model. It follows that we cannot build a *single* canonical model to realize all the consistent sets of $\mathbf{L_{Kh}}$ formulas simultaneously. Instead, for each maximal consistent set of $\mathbf{L_{Kh}}$ formulas we build a separate canonical model.

Definition 4 Given a maximal consistent set Γ w.r.t. $\mathbb{SKH}$, let $\Sigma_\Gamma = \{\langle\psi, \varphi\rangle \mid \mathcal{K}h(\psi, \varphi) \in \Gamma\}$, the canonical model for Γ is $\mathcal{M}^c_\Gamma = \langle \mathcal{S}^c_\Gamma, \mathcal{R}^c, \mathcal{V}^c\rangle$ where:

- $\mathcal{S}^c_\Gamma = \{\Delta \mid \Delta$ is a maximal consistent set w.r.t. $\mathbb{SKH}$ and $\Gamma|_{\mathcal{K}h} = \Delta|_{\mathcal{K}h}\}$;
- $\mathcal{R}^c : \Sigma_\Gamma \to 2^{\mathcal{S}^c \times \mathcal{S}^c}$ such that $\Delta \xrightarrow{\langle\psi,\varphi\rangle}_c \Theta$ iff $\mathcal{K}h(\psi, \varphi) \in \Gamma$, $\psi \in \Delta$, and $\varphi \in \Theta$;
- $p \in \mathcal{V}^c(\Delta)$ iff $p \in \Delta$.

Clearly Γ is a state in $\mathcal{M}^c_\Gamma$. We say that $\Delta \in \mathcal{S}^c_\Gamma$ is a φ-*state* if $\varphi \in \Delta$.

The following two propositions are immediate:

Proposition 4 *For any Δ, Δ' in $\mathcal{S}^c_\Gamma$, any $\mathcal{K}h(\psi, \varphi) \in \mathbf{L_{Kh}}$, $\mathcal{K}h(\psi, \varphi) \in \Delta$ iff $\mathcal{K}h(\psi, \varphi) \in \Delta'$ iff $\mathcal{K}h(\psi, \varphi) \in \Gamma$.*

Proposition 5 *For any $\Delta \in \mathcal{S}^c_\Gamma$, if $\Delta \xrightarrow{\langle\psi,\varphi\rangle} \Theta$ for some $\Theta \in \mathcal{S}^c_\Gamma$ then $\Delta \xrightarrow{\langle\psi,\varphi\rangle} \Theta'$ for any Θ' such that $\varphi \in \Theta'$.*

Now we prove a crucial proposition to be used repeatedly later.

Proposition 6 *If $\varphi \in \Delta$ for all $\Delta \in \mathcal{S}^c_\Gamma$ then $\mathcal{U}\varphi \in \Delta$ for all $\Delta \in \mathcal{S}^c_\Gamma$.*

Proof Suppose $\varphi \in \Delta$ for all $\Delta \in \mathcal{S}^c_\Gamma$, then by the definition of $\mathcal{S}^c_\Gamma$, $\neg\varphi$ is not consistent with $\Gamma|_{\mathcal{K}h} \cup \Gamma|_{\neg\mathcal{K}h}$, for otherwise $\Gamma|_{\mathcal{K}h} \cup \Gamma|_{\neg\mathcal{K}h} \cup \{\neg\varphi\}$ can be extended into a maximal consistent set in $\mathcal{S}^c_\Gamma$ due to a standard Lindenbaum-like argument, which contradicts the assumption that $\varphi \in \Delta$ for all $\Delta \in \mathcal{S}^c_\Gamma$. Thus there are $\mathcal{K}h(\psi_1, \varphi_1)$, …, $\mathcal{K}h(\psi_k, \varphi_k) \in \Gamma|_{\mathcal{K}h}$ and $\neg\mathcal{K}h(\psi'_1, \varphi'_1)$, …, $\neg\mathcal{K}h(\psi'_l, \varphi'_l) \in \Gamma|_{\neg\mathcal{K}h}$ such that

$$\vdash \bigwedge_{1\le i\le k} \mathcal{K}h\left(\psi_i, \varphi_i\right) \wedge \bigwedge_{1\le j\le l} \neg\mathcal{K}h\left(\psi'_j, \varphi'_j\right) \to \varphi.$$

By NECU,

$$\vdash \mathcal{U}\left(\bigwedge_{1\le i\le k} \mathcal{K}h(\psi_i, \varphi_i) \wedge \bigwedge_{1\le j\le l} \neg\mathcal{K}h\left(\psi'_j, \varphi'_j\right) \to \varphi\right).$$

By DISTU we have:

$$\vdash \mathcal{U}\left(\bigwedge_{1\le i\le k} \mathcal{K}h(\psi_i,\varphi_i)\wedge \bigwedge_{1\le j\le l} \neg\mathcal{K}h\left(\psi'_j,\varphi'_j\right)\right)\to \mathcal{U}\varphi.$$

Since $\mathcal{K}h(\psi_1,\varphi_1),\ldots,\mathcal{K}h(\psi_k,\varphi_k)\in\Gamma|_{\mathcal{K}h}$, we have $\mathcal{U}\mathcal{K}h(\psi_1,\varphi_1),\ldots,\mathcal{U}\mathcal{K}h(\psi_k,\varphi_k)\in\Gamma$ due to 4U and the fact that Γ is a maximal consistent set. Similarly, we have $\mathcal{U}\neg\mathcal{K}h(\psi'_1,\varphi'_1),\ldots,\mathcal{U}\neg\mathcal{K}h(\psi'_j,\varphi'_j)\in\Gamma$ due to 5U. Now due to a slight generalization of UCONJ, we have:

$$\mathcal{U}\left(\bigwedge_{1\le i\le k} \mathcal{K}h(\psi_i,\varphi_i)\wedge \bigwedge_{1\le j\le l} \neg\mathcal{K}h\left(\psi'_j,\varphi'_j\right)\right)\in\Gamma.$$

Now it is immediate that $\mathcal{U}\varphi\in\Gamma$. Due to Proposition 4, $\mathcal{U}\varphi\in\Delta$ for all $\Delta\in\mathcal{S}^c_\Gamma$. □

Now we are ready to establish another key proposition for the truth lemma.

Proposition 7 *If there are $\psi',\varphi'\in\mathbf{L_{Kh}}$ such that for each ψ-state $\Delta\in\mathcal{S}^c_\Gamma$ we have $\Delta\xrightarrow{\langle\psi',\varphi'\rangle}\Theta$ for some $\Theta\in\mathcal{S}^c_\Gamma$, then $\mathcal{U}(\psi\to\psi')\in\Delta$ for all $\Delta\in\mathcal{S}^c_\Gamma$.*

Proof Suppose that every ψ-state has an outgoing $\langle\psi',\varphi'\rangle$-transition, then by the definition of $\mathcal{R}^c$, ψ' is in all the ψ-states. For each Δ, either $\psi\notin\Delta$, or $\psi\in\Delta$. In the latter $\psi'\in\Delta$ follows. Now by the fact that Δ is maximally consistent it is not hard to show $\psi\to\psi'\in\Delta$ in both cases. By Proposition 6, $\mathcal{U}(\psi\to\psi')\in\Delta$ for all $\Delta\in\mathcal{S}^c_\Gamma$. □

Now we are ready to prove the truth lemma.

Lemma 1 (Truth lemma) *For any $\varphi\in\mathbf{L_{Kh}}$: $\mathcal{M}^c_\Gamma,\Delta\vDash\varphi\iff\varphi\in\Delta$.*

Proof Boolean cases are trivial and we only focus on the case of $\mathcal{K}h(\psi,\varphi)$.
$\Longrightarrow$: If $\mathcal{M}^c_\Gamma,\Delta\vDash\mathcal{K}h(\psi,\varphi)$, then according to the semantics, there exists a (possibly empty) sequence $\sigma\in\Sigma_\Gamma{}^*$ such that for every $\Delta'\vDash\psi$: σ is strongly executable at Δ' and if $\Delta'\xrightarrow{\sigma}\Delta''$ then $\mathcal{M}^c_\Gamma,\Delta''\vDash\varphi$. Due to the construction of $\mathcal{R}^c$, there are $\mathcal{K}h(\psi_1,\varphi_1),\ldots,\mathcal{K}h(\psi_k,\varphi_k)$ in Γ such that $\sigma=\langle\psi_1,\varphi_1\rangle\ldots\langle\psi_k,\varphi_k\rangle$ for some $k\ge 0$.

If there is no ψ-state, then by IH, $\neg\psi\in\Theta$ for all $\Theta\in\mathcal{S}^c_\Gamma$. By Proposition 6, $\mathcal{U}\neg\psi\in\Delta$, i.e., $\mathcal{K}h(\psi,\bot)\in\Delta$. Since $\bot\to\varphi$ and $\psi\to\psi$ are tautologies, by NECU, $\mathcal{U}(\bot\to\varphi)$ and $\mathcal{U}(\psi\to\psi)$ are also in Δ. Then by SUB and UKh $\mathcal{K}h(\psi,\varphi)\in\Delta$.

In the following we suppose that there exists some ψ-state and call this assumption ($\circ$). There are two cases to be considered:

- Suppose that $k=0$, i.e, $\sigma=\epsilon$: by the semantics, we have for any $\Theta\in\mathcal{S}^c_\Gamma$: $\Theta\vDash\psi\to\varphi$, i.e., $\Theta\nvDash\psi$ or $\Theta\vDash\varphi$. By IH, $\psi\notin\Theta$ or $\varphi\in\Theta$. Since Θ is maximally consistent, $\psi\to\varphi\in\Theta$ for all $\Theta\in\mathcal{S}^c_\Gamma$. By Proposition 6, $\mathcal{U}(\psi\to\varphi)\in\Theta$ for all $\Theta\in\mathcal{S}^c_\Gamma$. By SUB and EMPKh, we have $\mathcal{K}h(\psi,\varphi)\in\Theta$ for all $\Theta\in\mathcal{S}^c_\Gamma$, in particular, $\mathcal{K}h(\psi,\varphi)\in\Delta$.

- Suppose $k > 0$, recall that σ_m is the initial segment of σ up to $\langle \psi_m, \varphi_m \rangle$. We first prove the following claim ($\star$):
  For any $m \in \{1, \ldots, k\}$: (1) $\mathcal{K}h(\psi, \varphi_m) \in \Gamma$, and (2) each φ_m-state is reached via σ_m from every ψ-state.
  - $m = 1$: Due to the semantics of $\mathcal{K}h$, each ψ-state has an outgoing $\langle \psi_1, \varphi_1 \rangle$-transition. By Proposition 5, each ψ-state can reach all the φ_1 states by $\langle \psi_1, \varphi_1 \rangle$-transitions, which proves (2). By Proposition 7, $\mathcal{U}(\psi \to \psi_1) \in \Gamma$. By the definition of $\mathcal{R}^c$ it is clear that $\mathcal{K}h(\psi_1, \varphi_1) \in \Gamma$. Now by `SUB` and `UKh`, $\mathcal{K}h(\psi, \varphi_1) \in \Gamma$.
  - Suppose that the claim holds for $m = n-1$ then we prove that it holds for $m = n$ as well. By IH for the above claim ($\star$), we have (i) $\mathcal{K}h(\psi, \varphi_{n-1}) \in \Gamma$ and (ii) all the φ_{n-1}-states are reached from each ψ-state by σ_{n-1}. Since σ is a witness of the truth of $\mathcal{K}h(\psi, \varphi)$, σ is strongly executable on each ψ-state. Now due to (ii) and the strong executability of σ, we have (iii): each φ_{n-1}-state has some $\langle \psi_n, \varphi_n \rangle$-successor. By Proposition 7 we have $\mathcal{U}(\varphi_{n-1} \to \psi_n) \in \Gamma$. By the definition of $\mathcal{R}^c$, $\mathcal{K}h(\psi_n, \varphi_n) \in \Gamma$, thus by `UKh` we have $\mathcal{K}h(\varphi_{n-1}, \varphi_n) \in \Gamma$. Due to (i) and `COMPKh`, $\mathcal{K}h(\psi, \varphi_n) \in \Gamma$. Now for (2) of the claim: by (iii) and Proposition 5, each φ_n-state is reached from each φ_{n-1}-state via $\langle \psi_n, \varphi_n \rangle$. Thus based on (ii) again, we have each φ_n-state is reached from each ψ-state.

  Now Claim ($\star$) is proved. Let $m = k$, we have (1_k) $\mathcal{K}h(\psi, \varphi_k) \in \Gamma$ and (2_k) each φ_k-state is reached via $\sigma_k = \sigma$ from each ψ-state (under the assumption ($\circ$)). Now since σ witnesses the truth of $\mathcal{K}h(\psi, \varphi)$, $\mathcal{M}^c_\Gamma, \Delta' \vDash \varphi$ for every φ_k-state Δ'. By IH of the main structural induction over formulas, $\varphi \in \Delta'$ for every Δ' such that $\varphi_k \in \Delta'$. Thus it is not hard to see that $\varphi_k \to \varphi$ is in every state of $\mathcal{S}^c_\Gamma$, for otherwise there is a state Δ' such that $\varphi_k, \neg\varphi \in \Delta'$. By Proposition 6, $\mathcal{U}(\varphi_k \to \varphi) \in \Gamma$. Thus $\mathcal{K}h(\psi, \varphi) \in \Gamma$ due to (1_k), `COMPKh` and `SUB`. Therefore $\mathcal{K}h(\psi, \varphi) \in \Delta$ due to Proposition 4.

This completes the proof of $\mathcal{K}h(\psi, \varphi) \in \Delta$ if $\mathcal{M}^c_\Gamma, \Delta \vDash \mathcal{K}h(\psi, \varphi)$.
Now for the other way around:
$\Longleftarrow$: Suppose that $\mathcal{K}h(\psi, \varphi) \in \Delta$, i.e., $\mathcal{K}h(\psi, \varphi)$ is in all the states of $\mathcal{M}^c_\Gamma$ by Proposition 4, we need to show that $\mathcal{M}^c_\Gamma, \Delta \vDash \mathcal{K}h(\psi, \varphi)$. There are three cases to be considered.

- There is no Θ such that $\psi \in \Theta$. By IH, there is no Θ such that $\mathcal{M}^c_\Gamma, \Theta \vDash \psi$ then $\mathcal{M}^c_\Gamma, \Delta \vDash \mathcal{K}h(\psi, \varphi)$ trivially holds (by letting $\sigma = \epsilon$ as a vacuous witness).
- There are Θ, Θ' such that $\psi \in \Theta$ and $\varphi \in \Theta'$. Then by IH, we have $\mathcal{M}^c_\Gamma, \Theta \vDash \psi$ and $\mathcal{M}^c_\Gamma, \Theta' \vDash \varphi$ for such Θ and Θ'. Then by the construction of $\mathcal{R}^c$ and IH again we know that $\langle \psi, \varphi \rangle \in \Sigma_\Gamma$ is strongly executable, and it will take you to states where φ is true from any state where ψ is true.
- There is some Θ such that $\psi \in \Theta$ but no Θ such that $\varphi \in \Theta$. Then it is clear that $\neg\varphi \in \Theta$ for all $\Theta \in \mathcal{S}^c_\Gamma$. By Proposition 6, $\mathcal{U}\neg\varphi \in \Theta$ for all $\Theta \in \mathcal{S}^c_\Gamma$. Now we have $\mathcal{K}h(\varphi, \bot)$ and $\mathcal{K}h(\psi, \varphi) \in \Theta$ thus by `COMPKh` $\mathcal{K}h(\psi, \bot) \in \Theta$ namely $\mathcal{U}\neg\psi \in \Theta$. By `TU`, $\neg\psi \in \Theta$ for all the $\Theta \in \mathcal{S}^c_\Gamma$. However, this is contradictory to the assumption that $\psi \in \Theta$ for some $\Theta \in \mathcal{S}^c_\Gamma$.

$\square$

Now due to a standard Lindenbaum-like argument, each $\mathbb{SKH}$-consistent set of formulas can be extended to a maximal consistent set Γ. Due to the truth lemma, $\mathcal{M}^c_\Gamma, \Gamma \vDash \Gamma$. The completeness of $\mathbb{SKH}$ follows immediately.

Theorem 1 $\mathbb{SKH}$ *is sound and strongly complete w.r.t. the class of all models.*

Based on the canonical model construction it is easy to show that $\mathbf{L_{Kh}}$ has a small model property.

Proposition 8 *If a* $\mathbf{L_{Kh}}$ *formula* φ *is satisfiable then it has a model* $\mathcal{M}$ *such that* $|\mathcal{M}| \leq 2^n$ *where n is the number of proposition letters occurring in* φ*. It follows that* $\mathbf{L_{Kh}}$ *is decidable since it has a finite axiomatization.*

Proof Note that given a satisfiable formula φ, only the proposition letters that occur in φ matter. Thus we can consider a fragment of $\mathbf{L_{Kh}}$ based on the finite set of proposition letters in φ. Clearly, if φ is satisfiable in some model w.r.t. the full set of proposition letters $\mathbf{P}$ then it is satisfiable in a model w.r.t. the restricted set of proposition letters: we can simply forget the valuation of other propositions. Here comes the crucial observation that $\mathcal{K}h$-formulas hold globally in the canonical model. It follows that in the canonical model construction for the restricted language, the maximal consistent sets are essentially different valuations of the basic propositions in φ. Clearly, given the number of proposition letters n, the maximal size of the canonical model is 2^n. □

Similar to the modal logic $\mathbb{S}5$, we believe that the canonical model can be squeezed further by a proper selection of relevant parts, and leave the exact complexity to a further occasion.

4 Achieving while maintaining

"All is fair in love and war", but in many other cases how we achieve our goals matters a lot. For example, we do not want to win a game by playing dirty or to be rich by breaking the law: It is important to maintain our dignity. More generally, actions have costs, both financially or morally, we need to stay within our "budget" in reaching our goals. As an everyday life example, you may need to consider how much money you have in deciding how to go to the airport. Reaching your goal while maintaining some conditions is very important. In this section, we bring this explicitly into our language by extending the binary modality $\mathcal{K}h$ into a ternary one.

Definition 5 Given a set of proposition letters $\mathbf{P}$, the language $\mathbf{L_{Khm}}$ is defined as follows:

$$\varphi \ ::= \ \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathcal{K}h(\varphi, \varphi, \varphi)$$

where $p \in \mathbf{P}$. $\mathcal{K}h(\psi, \chi, \varphi)$ expresses that the agent knows how to achieve φ given ψ while maintaining χ strictly in-between (excluding the start and the end states).

Of course we can include the start and the end by $\mathcal{K}h(\psi \wedge \chi, \chi, \varphi \wedge \chi)$. Note that the binary $\mathcal{K}h(\psi, \varphi)$ can be expressed by $\mathcal{K}h(\psi, \top, \varphi)$. Similarly $\mathcal{U}\psi$ is now defined by $\mathcal{K}h(\neg\psi, \top, \bot)$.

The model for $\mathbf{L_{Khm}}$ stays the same but we need to generalize the notion of strongly executability w.r.t. a formula χ: we say that σ is *strongly* χ*-executable* at s if: $\sigma = \epsilon$, or $\sigma = a_1 \dots a_n$ and for any $0 \leq k < n$ and any t, $s \xrightarrow{\sigma_k} t$ implies that t has at least one a_{k+1}-successor, and if $k + 1 \neq n$ then all the a_{k+1} successors satisfy χ (w.r.t. the following semantics). The extra condition is clearly to guarantee χ in-between the start and the end.

Definition 6 (Semantics of $\mathbf{L_{Khm}}$)

$$\begin{array}{|lcl|}\hline \mathcal{M}, s \vDash \mathcal{K}h(\psi, \chi, \varphi) & \Leftrightarrow & \text{there exists a } \sigma \in \Sigma^* \text{ such that for all } s' \text{ such that } \mathcal{M}, s' \vDash \psi : \\ & & \sigma \text{ is strongly } \chi\text{-executable at } s' \text{ and for all } t \text{ such that } s' \xrightarrow{\sigma} t, \mathcal{M}, t \vDash \varphi \\ \hline \end{array}$$

Under this semantics, an interesting formula is $\mathcal{K}h(\psi, \bot, \varphi)$ which expresses that given ψ, φ can be reached in at most one step. Similarly, we can use $\mathcal{K}h(\psi, \bot, \varphi) \land \neg \mathcal{U}(\psi \to \varphi)$ to force the plan to be one-step precisely.

Clearly, our previous axioms still hold under the translation of $\mathcal{K}h(\psi, \varphi) := \mathcal{K}h(\psi, \top, \varphi)$. However we have further interesting axioms:

`COMPKhm`	$\mathcal{K}h(p, o, r) \land \mathcal{K}h(r, o, q) \land \mathcal{U}(r \to o) \to \mathcal{K}h(p, o, q)$
`EMPKhm`	$\mathcal{U}(p \to q) \to \mathcal{K}h(p, \bot, q)$
`ONEKhm`	$\mathcal{K}h(p, o, q) \land \neg \mathcal{K}h(p, \bot, q) \to \mathcal{K}h(p, \bot, o)$
`UKhm`	$\mathcal{U}(p' \to p) \land \mathcal{U}(o \to o') \land \mathcal{U}(q \to q') \land \mathcal{K}h(p, o, q) \to \mathcal{K}h(p', o', q')$

`COMPKhm` is a revised version of `COMPKh` where we need to combine the intermediate conditions. `EMPKhm` is a generalization of `EMPKh`. `ONEKhm` tells us that if the plan is more than one step then at least we need to make sure that we can make the first step to some intermediate states. Finally `UKhm` is a generalized version of `UKh` taking care of the weakening of the intermediate constraint. It is shown by Li and Wang (2017) that the above axioms plus `DISTU`, `TU` and the ternary versions of `4KU` and `5KU` are indeed enough to completely axiomatize the logic.

Another intended reason to introduce the middle condition in the $\mathcal{K}h$ modality is to have better control on the executions of the plans, in hopes of encoding certain dynamics of knowledge. Our treatment is inspired by van Benthem et al. (2006), where the *relativized common knowledge* operator was introduced to facilitate reduction axioms for epistemic logic with both common knowledge and public announcement. Let us first introduce the announcement-like modality into the language **PALKhm**:

$$\varphi \ ::= \ \top \mid p \mid \neg\varphi \mid (\varphi \land \varphi) \mid \mathcal{K}h(\varphi, \varphi, \varphi) \mid [\varphi]\varphi$$

Intuitively, $[\theta]\varphi$ says that φ holds after the information θ is provided. The update of the new information amounts to the change of the background knowledge throughout the model, and this will affect the knowledge-how. For example, a doctor may not know how to treat a patient since he is worried that the only available medicine may cause some very bad side-effect. Suppose a new scientific discovery shows that the side-effect is not possible under the relevant circumstance, then the doctor should know how to treat the patient. We define the semantics for $[\theta]$ following Plaza (1989):

Definition 7 (Semantics for **PALKhm**)

$$\boxed{\mathcal{M}, s \vDash [\theta]\varphi \Leftrightarrow \text{If } \mathcal{M}, s \vDash \theta \text{ then } \mathcal{M}|_\theta, s \vDash \varphi}$$

where $\mathcal{M}|_\theta$ is $\langle \mathcal{S}', \mathcal{R}', \mathcal{V}' \rangle$ with $\mathcal{S}' = \{t \in \mathcal{S}_\mathcal{M} \mid \mathcal{M}, t \vDash \theta\}$, $\mathcal{R}'(a) = \mathcal{R}(a)|_{\mathcal{S}' \times \mathcal{S}'}$ and $\mathcal{V}' = \mathcal{V}|_{\mathcal{S}'}$.

For example, the left model below depicts the situation where a doctor is not sure about the effect of the medicine a. Now if he is informed that the situation r can never happen under the current circumstance ($[\neg r]$), the model becomes the one at the right hand side below.

Example 3

$$\begin{tikzcd} s_1 : p \arrow[r, "a"] \arrow[dr, "a"] & s_3 : q & {[\neg r]} & s_1 : p \arrow[r, "a"] & s_3 : q \\ & s_2 : r & & & \end{tikzcd}$$

According to the semantics, $\neg\mathcal{K}h(p, \top, q) \wedge [\neg r]\mathcal{K}h(p, \top, q)$ holds at s_1 of the left model above.

van Benthem et al. (2006) showed that the announcement operator cannot be reduced to epistemic logic with common knowledge. Here the story is similar: it is impossible to reduce $[\theta]$ within $\mathbf{L_{Kh}}$ since we cannot control the process of the plan and it may go via some non-θ worlds. With the help of the middle condition, we hope to control the whole process and we have the following standard reductions from **PALKhm** to $\mathbf{L_{Khm}}$:

$$\boxed{\begin{array}{c} [\theta]p \leftrightarrow (\theta \to p) \\ [\theta]\neg\varphi \leftrightarrow (\theta \to \neg[\theta]\varphi) \\ [\theta](\varphi \wedge \psi) \leftrightarrow ([\theta]\varphi \wedge [\theta]\psi) \end{array}}$$

However, none of the following "natural" candidates to reduce $[\theta]\mathcal{K}h$ is valid:

$$\boxed{\begin{array}{l} [\theta]\mathcal{K}h(\psi, \chi, \varphi) \leftrightarrow (\theta \to \mathcal{K}h(\langle\theta\rangle\psi, [\theta]\chi, [\theta]\varphi)) \\ [\theta]\mathcal{K}h(\psi, \chi, \varphi) \leftrightarrow (\theta \to \mathcal{K}h(\langle\theta\rangle\psi, [\theta]\chi, \langle\theta\rangle\varphi)) \\ [\theta]\mathcal{K}h(\psi, \chi, \varphi) \leftrightarrow (\theta \to \mathcal{K}h(\langle\theta\rangle\psi, \langle\theta\rangle\chi, [\theta]\varphi)) \\ [\theta]\mathcal{K}h(\psi, \chi, \varphi) \leftrightarrow (\theta \to \mathcal{K}h(\langle\theta\rangle\psi, \langle\theta\rangle\chi, \langle\theta\rangle\varphi)) \end{array}}$$

To see this, consider the following two models:

$$\begin{tikzcd} s_1 : p, q \arrow[r, "a"] \arrow[d, "a"] & s_4 & & t_1 : p, q \arrow[r, "a"] & t_4 \\ s_2 : p, q \arrow[loop below, "a"] & s_3 : p \arrow[l, "a"] & & t_2 : p, q \arrow[loop below, "a"] & t_3 : p \arrow[l, "a"] \end{tikzcd}$$

It is not hard to see that $s_1 \vDash [p]\mathcal{K}h(p, \top, q)$ but $t_1 \nvDash [p]\mathcal{K}h(p, \top, q)$. However, the formula $p \to \mathcal{K}h(\langle p\rangle p, [p]\top, \langle p\rangle q)$ is false on s_1, and $p \to \mathcal{K}h(\langle p\rangle p, [p]\top, [p]q)$

is true on t_1. This is because requiring $\langle p\rangle q$ as the updated goal is too strong, while requiring $[p]q$ is on the other hand too weak: all the worlds satisfy it. Also note that changing $[p]\top$ into $\langle p\rangle\top$ will not help either, since the trouble is not really about the intermediate condition. To our surprise, in fact, it is impossible to use $\mathbf{L_{Khm}}$ to encode the dynamic operator!

Proposition 9 **PALKhm** *is strictly more expressive than* $\mathbf{L_{Khm}}$.

Proof (Sketch) Recall that in the above two models $s_1 \vDash [p]\mathcal{K}h(p, \top, q)$ but $t_1 \nvDash [p]\mathcal{K}h(p, \top, q)$. However, we can show that s_1 and t_1 are actually indistinguishable in $\mathbf{L_{Khm}}$. We can do induction on the structure of the formulas to show that for each $n \in \{1, 2, 3, 4\}$, s_n and t_n satisfy exactly the same $\mathbf{L_{Khm}}$ formula. The only non-trivial case is for $\mathcal{K}h(\psi, \chi, \varphi)$. We only need to consider the subcase when ψ holds on s_1 (and t_1 by IH), for otherwise the link between s_1 and s_2 is irrelevant in the evaluation of $\mathcal{K}h(\psi, \chi, \varphi)$ due to the fact that there is no incoming arrow to s_1 or t_1. Note that since the $\mathcal{K}h$ formulas are global and the propositional valuations are the same on s_1, s_2, t_1, t_2, then ψ holds on s_1 and t_1 implies that it holds on all those four states. Now note that the only strongly executable plans on both s_1 and s_2 (and on both t_1 and t_2) are ϵ and a. If a is executed, then starting from s_1 and s_2 we will end up with the set $\{s_2, s_4\}$; starting from t_1 and t_2 we will end up with the corresponding set $\{t_2, t_4\}$. If ϵ is executed then the resulting sets of states will be $\{s_1, s_2\}$ and $\{t_1, t_2\}$ respectively. Based on IH, $\{s_2, s_4\}$ both satisfy φ iff $\{t_2, t_4\}$ both satisfy φ, and $\{s_1, s_2\}$ both satisfy φ iff $\{t_1, t_2\}$ both satisfy φ. A moment of reflection should confirm that $\mathcal{K}h(\psi, \chi, \varphi)$ holds in one model iff $\mathcal{K}h(\psi, \chi, \varphi)$ holds in the other model. □

Note that in the above proof, the intermediate conditions do not really play a role, due to the special structures of the models. Therefore the above proof can be adapted to show that the announcement operator is not reducible within $\mathbf{L_{Kh}}$. It is still an open problem to find an extension of $\mathbf{L_{Kh}}$ (and $\mathbf{L_{Khm}}$) which can pre-encode the announcements.

Of course, there are also other types of dynamics which are reasonable in the setting of "knowing how", e.g., adding or deleting the relations which amounts to the changes of abilities [cf. (Areces et al. 2015)].

5 Conclusions and future work

In this paper, we propose and study a modal logic of goal-directed "knowing how". The highlights of our framework are summarized below with connections to our earlier ideas on non-standard epistemic logics:

- The "knowing how" construction is treated as a single modality similar to our works on "knowing whether" and "knowing what" (Wang and Fan 2013, 2014; Fan et al. 2014, 2015).
- The $\mathcal{K}h$ operator is treated as a special conditional: being able to guarantee a goal *given* a precondition, inspired by the conditionalization in (Wang and Fan 2014).
- The *ability* involved is further interpreted as having a plan that never fails to achieve the goal under the precondition, inspired by our previous work on conformant planning (Yu et al. 2016).

- The semantics is based on labeled transition systems representing the agent's knowledge of his own abilities, inspired by the framework experimented by Wang (2015b). However, there is no action symbol in our language.
- We also introduced the middle condition to express knowing how to achieve something while maintaining some conditions, which is inspired by relativized common knowledge operator proposed by van Benthem et al. (2006).
- The standard semantics of epistemic logic checks *all* the indistinguishable alternatives. However, our semantics for $\mathcal{K}h$ has a more existential flavor: knowing how as having at least *one* good plan.[17] Our modal operator is not *local* to the epistemic alternatives but it is about all the possible states even when they are distinguishable from the current world by the agent. Thus a cook can still be said to know how to cook a certain dish even if he knows that the ingredients are not available right now.

There are a lot more to explore in terms of model theory, proof theory and complexity analysis in order to thoroughly understand our logic. Moreover, it is a natural extension to introduce the standard "knowing that" operator $\mathcal{K}$ into the language, and correspondingly add a set $\mathcal{E} \subseteq \mathcal{S}$ in the model to capture the agent's *local* epistemic alternatives. Then we can define the local version of "knowing how" $\mathcal{K}h\varphi$ as $\mathcal{K}\psi \wedge \mathcal{K}h(\psi, \varphi)$ for some ψ. Other obvious next steps include probabilistic and multi-agent versions of $\mathcal{K}h$. It also makes good sense to consider group notions of "knowing how" which may bring it closer to the framework of ATEL, where a group of agents may achieve a lot more together [cf. Ågotnes et al. (2015)].

An important next step is to consider *contingent-plans*, instead of our conformant ones, which are no longer linear but with conditional actions based on the observation and knowledge of the agent. Such plans make more sense when there are new observations during the execution of the plan. Syntactically we may consider program-based "knowing how" where branching plans and iterated plans are allowed. However, if we keep the logic language as simple as it is now, it may turns out that the we still have the same logic (valid formulas).

There are also interesting philosophical questions related to our formal theory. For example, a new kind of logical omniscience may occur: If there is indeed a good plan to achieve φ according to the agent's abilities then he knows how to achieve φ. Philosophically, it is also debatable whether the introspection axioms should hold for knowledge-how. Moreover, to the taste of philosophers, maybe an empty plan is not acceptable to witness knowledge-how, e.g., people would not say one knows how to digest (by consciously doing nothing). We can define a stronger modality $\mathcal{K}h^+(\psi, \varphi)$ as $\mathcal{K}h(\psi, \varphi) \wedge \neg\mathcal{U}(\psi \to \varphi)$ to rule out such cases.[18] Note that although $\mathcal{U}$ is definable by $\mathcal{K}h$ in our setting, it does not have the philosophical implication that knowledge-that is actually a subspecies of knowledge-how which strong anti-intellectulism would agree. Nevertheless, our axioms do tell us something about the interactions between

[17] This connects with the philosophical concept of knowledge as a strengthened notion based on justified true belief where the existence of a good justification suffices, cf. also justification logic proposed by Artemov (2008).

[18] The distinction between $\mathcal{K}h$ and $\mathcal{K}h^+$ is similar to the distinction between STIT and deliberative STIT.

“knowing how” and “knowing that”, e.g., UKh says that (global) knowledge-that may let us know better how to reach our goal.

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References

Ågotnes, T., Goranko, V., Jamroga, W., & Wooldridge, M. (2015). Knowledge and ability, chapter 11. In H. van Ditmarsch, J. Halpern, W. van der Hoek, & B. Kooi (Eds.), *Handbook of epistemic logic*. London: College Publications.

Alur, R., Henzinger, T., & Kupferman, O. (2002). Alternating-time temporal logic. *Journal of the ACM*, *49*, 672–713.

Areces, C., Fervari, R., & Hoffmann, G. (2015). Relation-changing modal operators. *Logic Journal of the IGPL*, *23*(4), 601–627.

Artemov, S. (2008). The logic of justification. *The Review of Symbolic Logic*, *1*(04), 477–513.

Belardinelli, F. (2014). Reasoning about knowledge and strategies: Epistemic strategy logic. In *Proceedings of SR*, *2014* (pp. 27–33)

Broersen, J. (2008). A logical analysis of the interaction between ‘obligation-to-do’ and ‘knowingly doing’. In *Proceedings of DEON*, *08* (pp. 140–154)

Broersen, J., & Herzig, A. (2015). Using STIT theory to talk about strategies. In J. van Benthem, S. Ghosh, & R. Verbrugge (Eds.), *Models of strategic reasoning: Logics, games, and communities* (pp. 137–173). New York: Springer.

Brown, M. A. (1988). On the logic of ability. *Journal of Philosophical Logic*, *17*(1), 1–26.

Carr, D. (1979). The logic of knowing how and ability. *Mind*, *88*(351), 394–409.

Ciardelli, I., & Roelofsen, F. (2015). Inquisitive dynamic epistemic logic. *Synthese*, *192*(6), 1643–1687.

Ciardelli, I., Groenendijk, J., & Roelofsen, F. (2013). Inquisitive semantics: A new notion of meaning. *Language and Linguistics Compass*, *7*(9), 459–476.

Egré, P. (2008). Question-embedding and factivity. *Grazer Philosophische Studien*, *77*(1), 85–125.

Fagin, R., Halpern, J., Moses, Y., & Vardi, M. (1995). *Reasoning about knowledge*. Cambridge, MA: MIT Press.

Fan, J., Wang, Y., & van Ditmarsch, H. (2014). Almost necessary. *Advances in Modal Logic*, *10*, 178–196.

Fan, J., Wang, Y., & van Ditmarsch, H. (2015). Contingency and knowing whether. *The Review of Symbolic Logic*, *8*, 75–107.

Fantl, J. (2008). Knowing-how and knowing-that. *Philosophy Compass*, *3*(3), 451–470.

Fitting, M., & Mendelsohn, R. L. (1998). *First-order modal logic. Synthese library*. New York: Springer.

Gochet, P. (2013). An open problem in the logic of knowing how. In J. Hintikka (Ed.), *Open problems in epistemology*. Helsinki: The Philosophical Society of Finland.

Gochet, P., & Gribomont, P. (2006). Epistemic logic. In D. M. Gabbay & J. Woods (Eds.), *Handbook of the history of logic* (Vol. 7). North Holland: Elsevier.

Goranko, V., & Passy, S. (1992). Using the universal modality: Gains and questions. *Journal of Logic and Computation*, *2*(1), 5–30.

Gu, T., & Wang, Y. (2016). “Knowing value” logic as a normal modal logic. *Advances in Modal Logic*, *11*, 362–381.

Harrah, D. (2002). The logic of questions. In D. Gabbay (Ed.), *Handbook of philosophical logic* (Vol. 8). New York: Springer.

Hart, S., Heifetz, A., & Samet, D. (1996). Knowing whether, knowing that, and the cardinality of state spaces. *Journal of Economic Theory*, *70*(1), 249–256.

Herzig, A. (2015). Logics of knowledge and action: Critical analysis and challenges. *Autonomous Agents and Multi-Agent Systems*, *29*(5), 719–753.

Herzig, A., & Troquard, N. (2006). Knowing how to play: Uniform choices in logics of agency. In *Proceedings of AAMAS*, *06* (pp. 209–216)

Herzig, A., Lorini, E., & Walther, D. (2013). Reasoning about actions meets strategic logics. In *Proceedings of LORI*, *2013* (pp. 162–175)
Hintikka, J. (1962). *Knowledge and belief: An introduction to the logic of the two notions*. Ithaca, NY: Cornell University Press.
Hintikka, J. (1989). Reasoning about knowledge in philosophy: The paradigm of epistemic logic. In *The logic of epistemology and the epistemology of logic, synthese library* (Vol. 200, pp. 17–35). Dordrecht: Springer
Hintikka, J. (2007). *Socratic epistemology: Explorations of knowledge-seeking by questioning*. Cambridge: Cambridge University Press.
van der Hoek, W., & Lomuscio, A. (2003). Ignore at your peril: Towards a logic for ignorance. In *Proceedings of AAMAS*, *03* (pp. 1148–1149).
Jamroga, W., & van der Hoek, W. (2004). Agents that know how to play. *Fundamenta Informaticae*, *63*(2–3), 185–219.
Kracht, M., & Wolter, F. (1997). Simulation and transfer results in modal logic: A survey. *Studia Logica*, *59*(1), 149–177.
Lau, T., Wang, Y. (2016). Knowing your ability. *The Philosophical Forum*, pp. 415–424
Lenzen, W. (1978). Recent work in epistemic logic. *Acta Philosophica Fennica*, *30*(2), 1–219.
Li, Y., Wang, Y. (2017). Achieving while maintaining: A logic of knowing how with intermediate constraints. In: *Proceedings of ICLA'17* (forthcoming)
McCarthy, J. (1979). First-order theories of individual concepts and propositions. *Machine Intelligence*, *9*, 129–147.
McCarthy, J., & Hayes, P. J. (1969). Some philosophical problems from the standpoint of artificial intelligence. In *Machine intelligence* (pp. 463–502). Edinburgh University Press
Moore, R. C. (1985). A formal theory of knowledge and action. In J. R. Hobbs & R. C. Moore (Eds.), *Formal theories of the commonsense world*. Norwood: Ablex Publishing Corporation.
Morgenstern, L. (1986). A first order theory of planning, knowledge, and action. In *Proceedings TARK 86* (pp. 99–114). San Francisco: Morgan Kaufmann Publishers Inc.
Noë, A. (2005). Against intellectualism. *Analysis*, *65*, 278–290.
Plaza, J. A. (1989). Logics of public communications. In M. L. Emrich, M. S. Pfeifer, M. Hadzikadic, Z. W. Ras (Eds.), *Proceedings of the 4th international symposium on methodologies for intelligent systems* (pp. 201–216)
Ryle, G. (1946). Knowing how and knowing that: The presidential address. *Proceedings of the Aristotelian Society*, *46*, 1–16.
Smith, D. E., & Weld, D. S. (1998). Conformant graphplan. In: *AAAI98* (pp. 889–896).
Stanley, J. (2011). *Know how*. Oxford: Oxford University Press.
Stanley, J. (2001). Knowing how. *The Journal of Philosophy*, *98*, 411–444.
Tillio, A. D., Halpern, J. Y., & Samet, D. (2014). Conditional belief types. *Games and Economic Behavior*, *87*, 253–268.
van Benthem, J., van Eijck, J., & Kooi, B. (2006). Logics of communication and change. *Information and Computation*, *204*(11), 1620–1662.
van Ditmarsch, H., Halpern, J. Y., van der Hoek, W., & Kooi, B. (2015). *Handbook of epistemic logic*. London: College Publications.
Vendler, Z. (1967). *Linguistics in philosophy*. Ithaca: Cornell University Press.
Von Wright, G. H. (1951). *An essay in modal logic*. Amsterdam: North Holland.
Wang, R. J. (2011). Timed modal epistemic logic. PhD thesis, City University of New York
Wang, Y. (2015a). A logic of knowing how. In *Proceedings of LORI*, *2015* (pp. 392–405)
Wang, Y. (2015b). Representing imperfect information of procedures with hyper models. In *Proceedings of ICLA*, *2015* (pp. 218–231)
Wang, Y. (forthcoming). Beyond knowing that: A new generation of epistemic logics. In H. van Ditmarsch & G. Sandu (Eds.), *Jaakko Hintikka on knowledge and game theoretical semantics*. New York: Springer.
Wang, Y., & Fan, J. (2013). Knowing that, knowing what, and public communication: Public announcement logic with Kv operators. In *Proceedings of IJCAI*, *13* (pp. 1147–1154)
Wang, Y., & Fan, J. (2014). Conditionally knowing what. *Advances in Modal Logic*, *10*, 569–587.
Wang, Y., & Li, Y. (2012). Not all those who wander are lost: Dynamic epistemic reasoning in navigation. *Advances in Modal Logic*, *9*, 559–580.
Yu, Q., Li, Y., & Wang, Y. (2016). A dynamic epistemic framework for conformant planning. In *Proceedings of TARK'15, EPTCS* (pp. 298–318)